Crucial role of strategy updating for coexistence of strategies in interaction networks

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Network models are useful tools for studying the dynamics of social interactions in a structured population. After a round of interactions with the players in their local neighborhood, players update their strategy based on the comparison of their own payoff with the payoff of one of their neighbors. Here we show that the assumptions made on strategy updating are of crucial importance for the strategy dynamics. In the first step, we demonstrate that seemingly small deviations from the standard assumptions on updating have major implications for the evolutionary outcome of two cooperation games: cooperation can more easily persist in a Prisoner's Dilemma game, while it can go more easily extinct in a Snowdrift game. To explain these outcomes, we develop a general model for the updating of states in a network that allows us to derive conditions for the steady-state coexistence of states (or strategies). The analysis reveals that coexistence crucially depends on the number of agents consulted for updating. We conclude that updating rules are as important for evolution on a network as network structure and the nature of the interaction.

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I. INTRODUCTION

Network theory has provided important insights into the dynamics of interactions in a structured population. In this framework, population structure is represented by a network, the nodes of which represent the individual agents while the links correspond to the possible interactions [1–4]. The agents can be molecules, individual organisms, or groups of individuals, and the interactions can also be highly diverse, ranging from chemical reactions among molecules to the exchange of goods or knowledge among groups [5–8]. To fix ideas, we will here focus on the evolution of social interactions among individuals. In this context, network models typically assume that each agent is endowed with a certain strategy (corresponding to the agent's "state") that determines the agent's behavior in interactions with their neighbors in the network and the resulting payoffs. After the interaction phase, agents can update their strategy by comparing their own accumulated payoff with the payoff of one of their neighbors [9–11].

Network models have revealed that network structure plays an important role for the evolutionary dynamics of behavior in a social interaction [11–14]. Take, for example, the Prisoner's Dilemma game (PDG) [15,16], where mutual cooperation is favored to mutual defection by both players. Yet, cooperation is outcompeted by defection in a well-mixed population, since defection is a dominant strategy. When interactions take place on a network, however, cooperation can get established, but this strongly depends on the network structure; cooperation gets easily off the ground in heterogeneous networks (e.g., scale-free networks), while it will not easily evolve in homogeneous networks (e.g. random-regular networks) [17,18]. In a Snowdrift game (SDG), another prototypical example for the evolution of cooperation, the coexistence of cooperation and defection is expected in a well-mixed population, while network models predict the fixation of either cooperation or defection under a wide range of conditions [19–21].

In addition to network structures, strategy updating can have important implications for evolution in an interaction network [14,22–25]. Here we will demonstrate that the coexistence of strategies in a network strongly depends on strategy updating.

II. TWO ILLUSTRATIVE EXAMPLES

For simplicity, we consider games with two pure strategies, like the PDG or the SDG. At each point in time, an agent employs one of the two strategies. The payoff obtained by an agent using strategy i in an interaction with an agent using strategy j is given by m_{ij}, where $M = (m_{ij})$ is the 2×2 payoff matrix characterizing the game. For example, the payoff matrices of a PDG and an SDG are given by

$$M_{\mathrm{PDG}} = \begin{pmatrix} b-c & -c \\ b & 0 \end{pmatrix}, \quad M_{\mathrm{SDG}} = \begin{pmatrix} b-\frac{c}{2} & b-c \\ b & 0 \end{pmatrix}, \tag{1}$$

where b and c ($b > c$) indicate the benefits and costs of cooperation, respectively. Typically, strategy updating is modeled as follows [9–11]: an agent having used strategy A and accumulated payoff π_A in the previous interactions randomly selects another agent from her neighborhood; if that agent happens to have used the alternative strategy B and accumulated payoff π_B, then the focal agent will switch from A to B with a probability $u_{A\to B}$ that reflects the payoff difference $\pi_B - \pi_A$. This probability may, for example, be given by the Fermi function

$$u_{A\to B} = (1 + e^{-\beta(\pi_B - \pi_A)})^{-1}. \tag{2}$$

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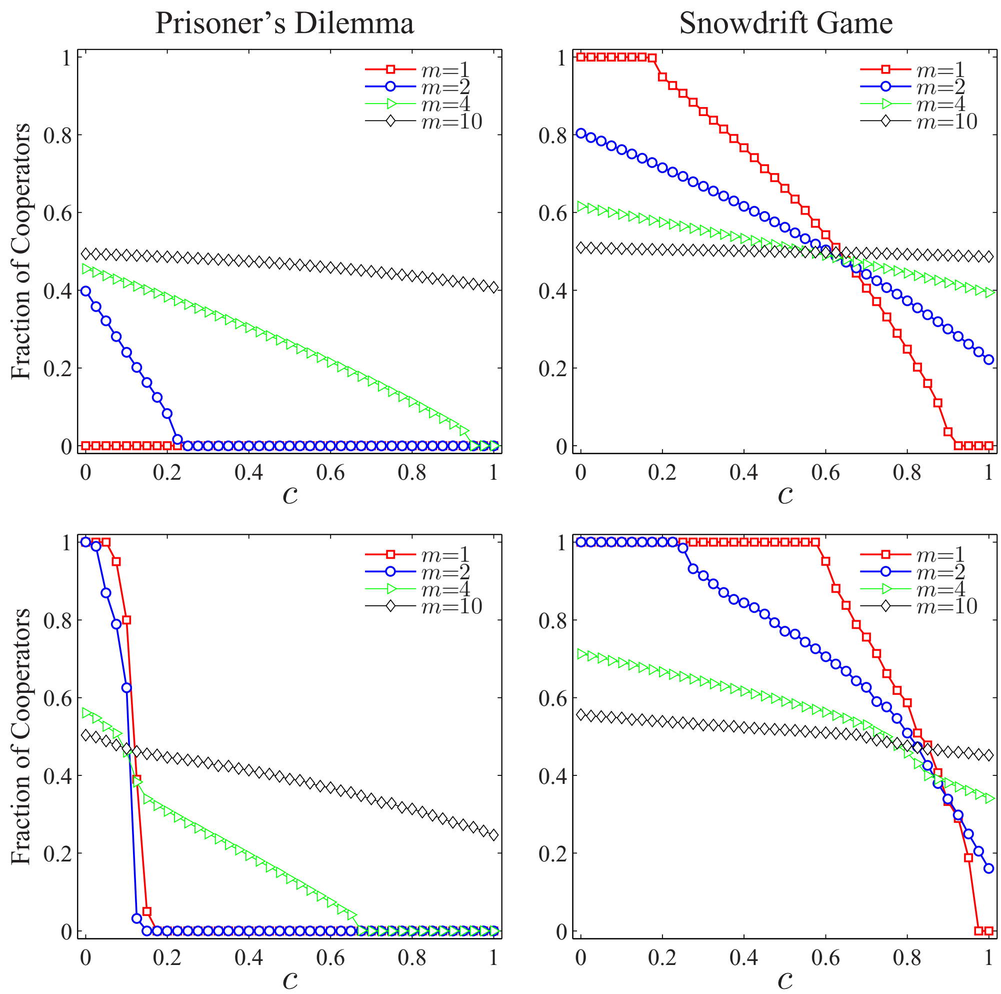

FIG. 1. (Color online) Equilibrium fraction of cooperators in the Prisoner's Dilemma game (PDG) and the Snowdrift game (SDG) as a function of the costs c of cooperation for four values of m, the number of agents consulted for strategy updating. Left panels: PDG; right panels: SDG; upper panels: random-regular network with degree 10; lower panel: Barabási-Albert scale-free network with average degree 10. The benefit of cooperation was kept constant at $b = 1$. In our simulations on scale-free networks, if a player's degree was smaller than m, she chose all of her neighbors for consultation.

The red curves in Fig. 1 show the evolution of cooperation in the PDG and the SDG for $b = 1$ and a spectrum of c values for this updating rule. The results confirm that cooperation in a PDG can evolve in a scale-free network (for $c < 0.1$) but not in a random-regular network, and that cooperation in the SDG will spread to fixation more easily in a scale-free network (for $c < 0.2$) than in a random-regular network (for $c < 0.6$). The other curves in Fig. 1 illustrate what happens if the strategy updating is not based on the consultation of one other agent, but on the consultation of two or more other agents. In these cases, a focal agent compares her payoff with that of m other agents and switches from A to B whenever any of these comparisons would result in such a switch in the standard updating scenario ($m = 1$) considered above.

Figure 1 clearly shows that such a change in strategy updating has a major effect on the evolutionary outcome. Now cooperation in the PDG can also get off the ground in a random-regular network ($m = 2$: $c < 0.2$; $m = 4$: $c < 0.9$; $m = 10$: all c). Most strikingly, for larger values of m, fixation for either cooperation or defection gives rise to the stable coexistence of these strategies. Moreover, for large values of m, the evolutionary outcome is relatively independent of the type of interaction (i.e., PDG versus SDG) and the structure of the network (i.e., random-regular versus scale-free).

We also considered still another updating rule: agents interact sequentially with their neighbors (in random sequence) and update their strategy as above, but now updating takes place after each individual interaction. In other words, the switching probability is given by (2), but now the payoffs of the A and the B players are not accumulated over several interactions, but given by the payoffs of a single interaction: $\pi_A = m_{AB}$ and $\pi_B = m_{BA}$. Many simulations for a large variety of payoff matrices M have revealed that, irrespective of the structure of the network, the evolutionary outcome is only dependent on the sign of $m_{12} - m_{21}$: if $m_{12} > m_{21}$, strategy 1 will spread to fixation; if $m_{12} < m_{21}$, strategy 2 will spread to fixation; and both strategies will coexist at equal frequencies if $m_{12} = m_{21}$. Hence, coexistence is very unlikely. But, again, we arrive at the conclusion that the evolutionary outcome is more strongly affected by the updating rule than by the nature of the interaction (which is dependent not only on the payoff parameters m_{12} and m_{21} but also on m_{11} and m_{22}) or the structure of the network.

III. GENERAL ANALYSIS

To explain the simulation results mentioned above, we now take a more mathematical approach that is applicable beyond the context of evolutionary games. This approach is based on the two transition probabilities $u_{A\to B}$ and $u_{B\to A}$, which are viewed as given model parameters that do not necessarily reflect a fitness comparison. As above, $u_{A\to B}$ denotes the probability that an agent using strategy A will switch to the alternative strategy B when this agent happens to consult a B player. Figure 2(a) shows the setup for the case $m = 1$ where a single updating takes place between two players with different strategies. Figure 2(b) represents a network scenario in which the focal player chooses m neighbors for updating. In this specific case, $m = 4$ and three of the four chosen neighbors maintain different strategies. If p_Ω denotes the probability that a neighbor of agent i uses strategy A, we can now calculate the probabilities $U^i_{A\to B}$ and $U^i_{B\to A}$ with which, after consulting m neighbors, agent i would switch from A to B or from B to A, respectively:

$$\begin{aligned} 1 - U^i_{A\to B} &= \left[1 - u_{A\to B}\left(1 - p_{\Omega_i}\right)\right]^m \\ 1 - U^i_{B\to A} &= \left(1 - u_{B\to A}\,p_{\Omega_i}\right)^m. \end{aligned} \tag{3}$$

For example, $u_{B\to A}\,p_{\Omega_i}$ is the probability that agent i, when having played B, is consulting an A-playing neighbor that induces agent i to switch to A; $1 - u_{B\to A}\,p_{\Omega_i}$ is the probability that any given neighbor does not induce agent i to switch when having played B, and $(1 - u_{B\to A}\,p_{\Omega_i})^m$ is the probability that none of m consulted neighbors will induce player i to switch to A. By definition, the latter probability corresponds to $1 - U^i_{B\to A}$.

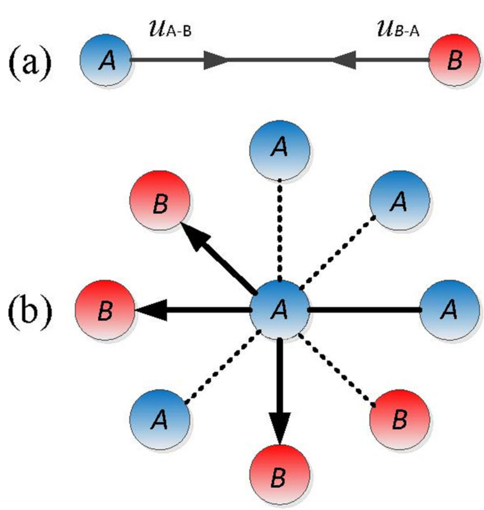

FIG. 2. (Color online) (a) A single updating when two different individuals encounter each other. Strategy A (blue) has a probability $u_{A\to B}$ to switch to B (red), while B switches to A with probability $u_{B\to A}$. (b) Diagrams illustrating an updating event in a network scenario where each player chooses m of her neighbors randomly for updating. Each arrow specifies a probabilistic switching because it is formed by different strategies. Dotted lines indicate neighbors which are not selected at the current time step.

We can now derive a recurrence equation for the probability $p_i(t)$ that a given agent i will employ strategy A at time t:

$$p_i(t+1) = p_i(t)\left[1 - U^i_{A\to B}(t)\right] + [1 - p_i(t)]U^i_{B\to A}(t). \tag{4}$$

The first term on the right-hand side corresponds to the joint probability of having played A in the previous time step and not having switched to B, while the second term corresponds to the probability of having played B at time t but having switched to A. An equilibrium $p_i(t+1) = p_i(t) = \hat{p}_i$ of (4) is characterized by

$$\hat{p}_i \cdot \hat{U}^i_{A\to B} = (1 - \hat{p}_i)\cdot \hat{U}^i_{B\to A}. \tag{5}$$

For a homogeneous network, such as a random-regular network, it is plausible to assume that the probability to use strategy A will converge to the same value $\hat{p}_i = \hat{p}_{\Omega_i} = \hat{p}_i$ for all i. Inserting (3) in (5) yields an implicit equation for $\hat{p}$:

$$\begin{aligned} &\hat{p}_i \cdot \{1 - [1 - u_{A\to B}(1 - \hat{p})]^m\} \\ &\quad = (1 - \hat{p})\cdot[1 - (1 - u_{B\to A}\hat{p})^m]. \end{aligned} \tag{6}$$

For $m = 1$, this immediately implies that equilibrium coexistence of both strategies (i.e., $0 < \hat{p} < 1$) is possible if, and only if, $u_{A\to B} = u_{B\to A}$. This explains our earlier results that strategy updating after each individual interaction will only lead to the coexistence of the two strategies if $m_{12} = m_{21}$. It also implies that in a homogeneous network strategy coexistence requires that, at equilibrium, both strategies have the same payoffs: $\pi_A(\hat{p}) = \pi_B(\hat{p})$.

Figure 3 illustrates that for $m > 1$ the coexistence of A and B is easy to achieve. For two values of $u_{A\to B}$, this figure shows the equilibrium frequency $\hat{p}$ of strategy A for a spectrum of values $u_{B\to A}$ and the outcome of simulations that are in excellent agreement with the equilibrium value predicted by (6). For a given value of $u_{A\to B}$, strategy A will persist in the population whenever $u_{B\to A}$ is larger than a certain threshold value $u^*_{B\to A}$. This minimum value for $\hat{p} > 0$ can be calculated by taking the limit $\hat{p} \to 0$ in (6).

When $u_{B\to A} \to u^*_{B\to A}$, the probability that a player i adopts strategy A is $\hat{p}_i$, where $0 < \hat{p}_i \ll 1$. Assume that after a long time evolution, $\hat{p}_i \approx \hat{p} \approx \hat{p}_{\Omega_i}$, then after substituting this into Eq. (6), and neglecting the high-order terms in $\hat{p}_i$, we get

$$m\hat{p}u^*_{B\to A} = \hat{p} - \hat{p}(1 - u_{A\to B})^m. \tag{7}$$

For fixed values of $u_{A\to B}$ and m, we have

$$u^*_{B\to A} = \frac{1 - (1 - u_{A\to B})^m}{m}. \tag{8}$$

Similarly, strategy B will persist for a given value of $u_{B\to A}$ whenever $u_{A\to B}$ is larger than a threshold value $u^*_{A\to B}$, which can be obtained from (6) by taking the limit $\hat{p} \to 1$. The result is

$$\begin{aligned} &\text{A will persist if } u_{B\to A} > \frac{1 - (1 - u_{A\to B})^m}{m} \\ &\text{B will persist if } u_{A\to B} > \frac{1 - (1 - u_{B\to A})^m}{m}. \end{aligned} \tag{9}$$

Coexistence of A and B will occur if both conditions are satisfied. Figure 4 illustrates that simulations in random-regular networks are in excellent agreement with this prediction and that the coexistence region becomes very large already for moderate values of m (e.g. $m = 4$).

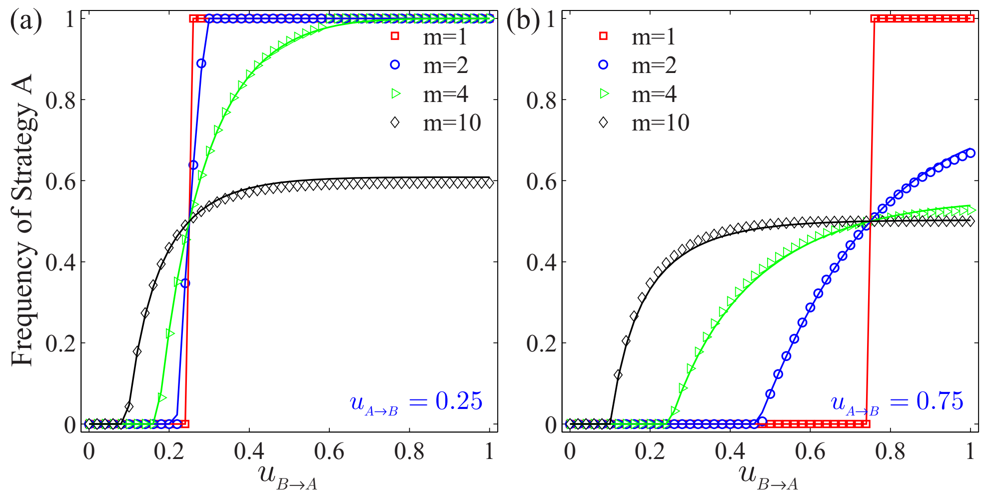

FIG. 3. (Color online) Equilibrium frequency of strategy A in a random-regular network of degree 10 for four values of m, the number of agents consulted for strategy updating. For (a) $u_{A\to B} = 0.25$ and (b) $u_{A\to B} = 0.75$, the analytical predictions based on Eq. (6) (solid lines) and the outcome of simulations (symbols) are shown for a spectrum of values of $u_{B\to A}$. Both panels clearly indicate that a larger value of m favors the equilibrium coexistence of both strategies. In our simulations time is discretized in time steps, and in each step players choose to be an A or a B player with the probability determined when finishing the previous step. We start from a configuration in which each player adopts strategy A with a probability chosen uniformly from the range [0, 1]. In each round, player i updates her strategy and is correspondingly associated with a probability that she is an A player in the next round. Each simulation result corresponds to a result of averaging over 10^3 generations after a transient period of 10^4 rounds in 100 independent realizations with the population size 10^4.

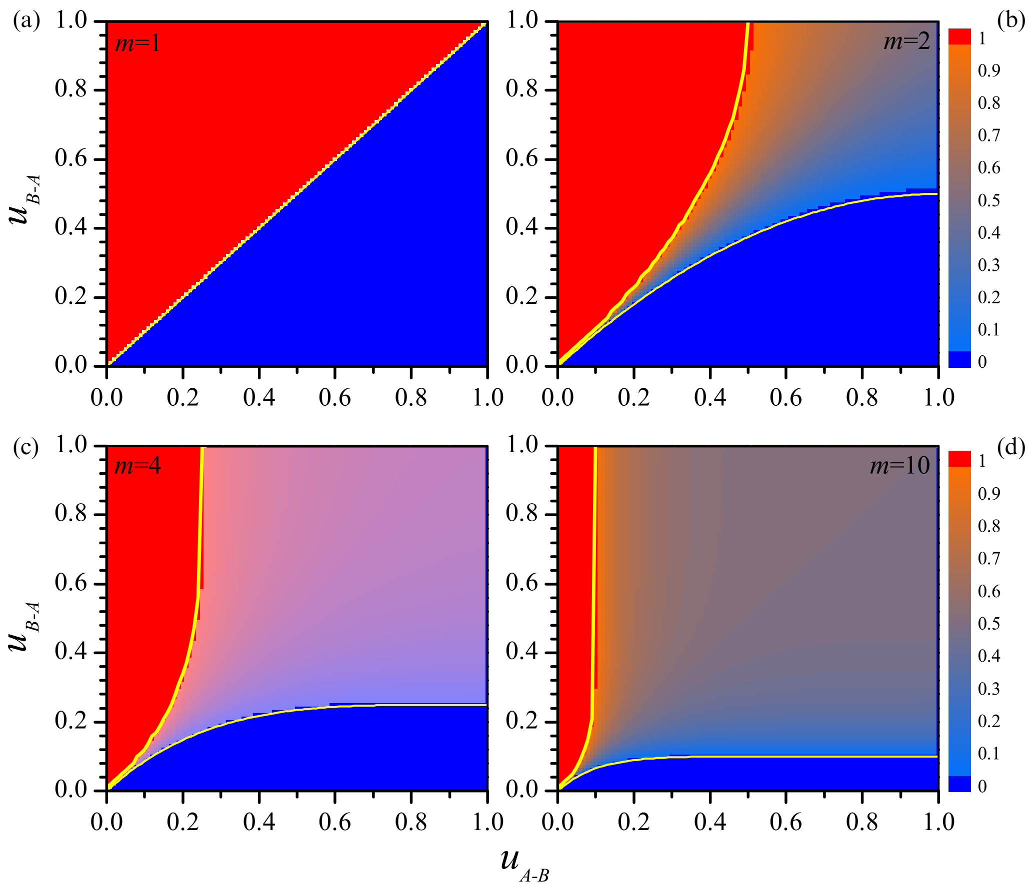

FIG. 4. (Color online) Equilibrium coexistence of strategies A and B as a function of the updating probabilities $u_{A\to B}$ and $u_{B\to A}$ for four values of m, the number of agents consulted for strategy updating: (a) $m = 1$; (b) $m = 2$; (c) $m = 4$; (d) $m = 10$. Red: fixation of strategy A; blue: fixation of strategy B; yellow lines: boundaries of coexistence region based on Eq. (7); all other colors: frequency of A ($0 < \hat{p} < 1$) resulting from Eq. (6).

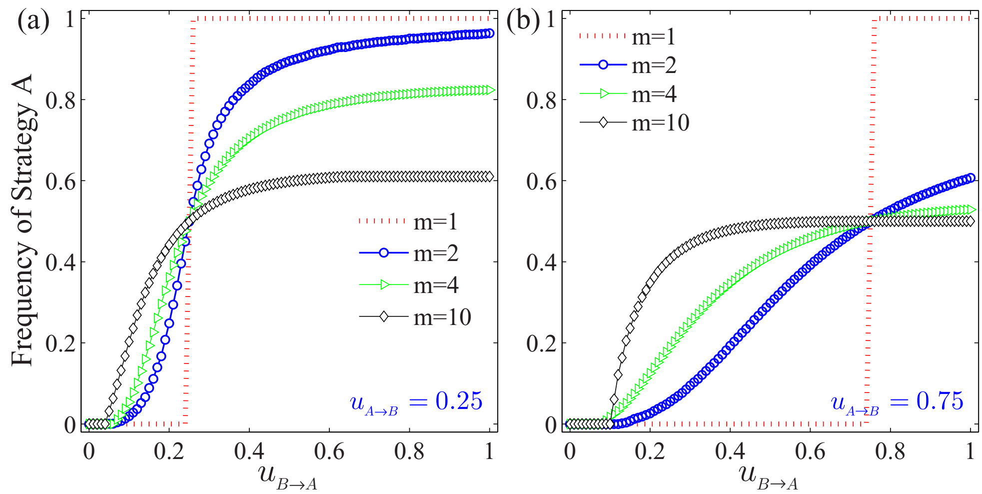

FIG. 5. (Color online) Equilibrium frequency of strategy A in a Barabási-Albert scale-free network with average degree 10 for four values of m, the number of agents consulted for strategy updating. For (a) $u_{A\to B} = 0.25$ and (b) $u_{A\to B} = 0.75$, the outcome of simulations is shown for a spectrum of values of $u_{B\to A}$. As in case of a random-regular network (Fig. 3), a larger value of m favors the equilibrium coexistence of both strategies.

IV. STRATEGY COEXISTENCE IN HETEROGENEOUS NETWORKS

Our analytical results do not directly apply to heterogeneous networks, since the equilibrium value $\hat{p}_i$ of the probability to use strategy A will depend on the degree of player i. As a rule, $\hat{p}_i$ will more likely be between 0 and 1 when the degree of player i is higher. Qualitatively, however, our basic insight that a larger value of m favors polymorphism for a broad range of values of $u_{A\to B}$ and $u_{B\to A}$ also applies to heterogeneous networks. This is illustrated by Fig. 5, which indicates for a Barabási-Albert scale-free network that the conditions for the coexistence of competing strategies are even less stringent than in a random-regular network. Here the critical values of $u_{B\to A}$ for a given $u_{A\to B}$ in scale-free networks are smaller than that in random-regular networks.

V. CONCLUSIONS

In conclusion, we have shown that evolution on an interaction network can be as strongly affected by the strategy updating procedure as by the network structure and the payoff matrix. In this paper for two-strategy evolutionary games in structured populations, we follow a different approach, bypassing the requirement for explicit knowledge of the exact payoffs, by encoding the payoffs into the willingness of any player to switch from her current strategy to the competing one. Theoretical computations and numerical simulations show that the evolutionary dynamics are intrinsically regulated by contact relationships specified by the network topologies of the populations. We demonstrate that updating rules are of crucial importance for the steady-state distribution of states. On the basis of general arguments, we show that the coexistence of different states strongly depends on the number m of agents that determine the updating of a given agent: if $m = 1$, as typically assumed, coexistence is difficult to achieve, while coexistence occurs under mild conditions when $m > 1$. By means of two cooperation games, we show that this general insight has important implications for the strategy dynamics of games on a network. In comparison to earlier models, cooperation can more easily persist in a Prisoner's Dilemma game, while it can go more easily extinct in a Snowdrift game. This implies that strategy updating deserves more attention in empirical and theoretical studies.

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