CogniGron Master Project: Compositionality of Memristor Networks

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Introduction

The classical elements of electrical circuits are resistors, capacitors, and inductors. These elements relate the four fundamental quantities current, voltage, charge, and flux-linkage. The existence of a fourth element, dubbed the *memristor* as a portmanteau of *memory resistor*, was postulated by Chua [5] as a relation between the charge and flux-linkage. The key feature of a memristor is that the instantaneous resistance value in a memristor depends on the history of its electrical stimulation. Therefore, memristors, or more generally materials with meristive properties, are considered as potential building blocks for neuromorphic computing systems.

The construction of neuromorphic computing systems is potentially achieved by creating *networks* of circuit elements. Series and parallel interconnections of memristors are studied in [4, 12]. For simulation studies of more complex network structures, see [11]. Networks that also include other elements, such as inductors or capacitors, can lead to richer dynamics. An example is given in [6], where the dynamics are considered for networks in which the inclusion of active elements leads to oscillatory behaviour. Also the synchronization of networks with memristors, which often play the role as adaptive coupling between neurons modelled as oscillators, has been studied [8].

However, memristor networks have not been studied systematically from a *compositional* point of view. How can the dynamics of a large network be deduced from the dynamics of its constituent parts? This project proposal aims to address this question using the language of category theory.

The mathematics of compositionality

The principle of compositionality asserts that the nature of complex structures is entirely determined by that of their simpler parts and the way these are composed. A framework for studying questions related to compositionality is provided by category theory which describes mathematical structures in terms of a labeled directed graph called a *category*. The nodes of this graph are called *objects* and the labelled directed edges are called *morphisms*. Each object is required to have an identity morphism and the composition of morphisms has to be associative. Some examples of categories are:

- the category of sets in which each object is a set and each morphism is a function;
- the category of groups in which each object is a group and each morphism is a group homomorphism;
- the category of vector spaces in which each object is a vector space and each morphism is a linear transformation.

Categories themselves can be interpreted as objects in the category of categories. The morphisms in the latter category are called *functors* and specify how objects and morphisms in a certain category transform to those in another one while preserving identities and composition. For a basic book on category theory aimed at a broad audience, see [9]. For an introductory text with a particular focus on the interconnection of systems, see [7].

Concepts within category theory, such as monoidal categories and decorated cospans, have proven to be very useful in studying so-called *open dynamical systems* which interact with their environment through inputs and outputs. In this context, compositionality can be understood as combining two systems by taking the output of one system as the input for another system.

Spivak [10] develops a technique for calculating the steady states of an interconnected system in terms of the steady states of its component dynamical systems. It is shown that the compositionality structure of dynamical systems fits with the monoidal structure of so-called *steady state matrices*. Serial, parallel, and feedback composition of matrices correspond to multiplication, Kronecker product, and partial trace operations, respectively.

Baez and Fong [2] study passive linear networks in the form of circuits made of resistors, inductors, and capacitors. A category is described in which a morphism is a circuit of this sort with marked input and output terminals. In addition, a *black box functor* is constructed that takes a circuit, forgets its internal structure, and only remembers its external behaviour. Two circuits have the same external behaviour if and only if they impose same relation between currents and potentials at their terminals.

In a companion paper [1], Baez et al. study electrical networks in terms of *props*, which are particular instances of strict symmetric monoidal categories. The objects in this category are natural numbers and the tensor product of objects is defined by addition. The behaviour of these circuits is described by in terms of morphisms between props. A new construction of the black box functor as given in [2] is provided which generalizes more easily to circuits with nonlinear components.

Baez and Pollard [3] treat open reaction networks as morphisms in a suitable category. Composing two such morphisms amounts to connecting the outputs of the first network to the inputs of the second network. A functor is constructed that sends any open reaction network to its corresponding open dynamical system. In addition, a black box functor is constructed that sends any open dynamical system to the relation that it imposes between input and output variables in steady states.

Aim of the project

Memristor models take the form of open dynamical systems which interact with their environment through external variables such as voltage or current. Therefore, connecting inputs with outputs of such models leads to networks of memristors. In addition, other circuit elements, such as capacitors or inductors, can be included as well. The general aim of this project is to study such networks from a compositional point of view.

Master students who would like to work on such a project should be willing to learn category theory (which is not typically offered in degree programmes at the University of Groningen). A good start would be reading the books by Fong and Spivak [7, 9]. The next step is to study the papers [10, 2, 1, 3] and references therein to obtain an understanding how category theory can be applied to study the interconnection of open dynamical systems.

The project described above is deliberately formulated in a broad way and offers subprojects that can be taken up by multiple students. Here is a list of possible subprojects that an interested student can work on (in order of increasing difficulty):

- Perform a thorough literature study and explain concepts like *monoidal categories*, *decorated cospans*, and *props*. In particular, explain how these concepts are used to study the composition of open dynamical systems. As a next step, write a proposal how such concepts can be used specifically for circuits that contain *nonlinear* circuit elements such as memristors.
- Develop a compositional framework for networks which only contain memristors. In particular, the aim is to construct a black box functor as in [3] which sends any meristor network to its corresponding open dynamical system.
- Develop a compositional framework for networks which contain both memristors and other circuit elements, such as capacitors (which may serve as models for neurons).

References

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