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Abstract

Motivated by yield curve modeling, we solve dynamic mean-variance efficiency problems in both discrete and continuous time. Our solution applies to both complete and incomplete markets and we do not require the existence of a riskless asset, which is relevant for yield curve modeling. Stochastic market parameters are incorporated using a vector of state variables. In particular for markets with deterministic parameters, we provide explicit solutions. In such markets, where no riskless asset need be present, we describe term-independent uniformly mean-variance efficient investment strategies. For constant parameters we show the existence of a unique, symmetrically distributed, trend stationary, uniformly MV efficient strategy.

Key words: Dynamic mean-variance analysis; Mean-variance portfolio selection; Efficient frontier; Mutual fund theorem

JEL classification: G11; G12; C61

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1 Introduction

Usually mean-variance (MV) analysis is applied for solving portfolio selection problems. The results on dynamic MV analysis presented here, however, are motivated by their application in the field of yield curve modeling. In fact, the price process of a riskless bond maturing at date T represents a dynamic MV efficient investment strategy, since the variance of the portfolio value at time T equals zero. It is therefore interesting to use MV properties of an underlying incomplete risky market to model the term structure of interest rates. In particular, Bekker and Bouwman (2009b) use risky assets as factors, and exploit the dynamic MV properties of the underlying risky market to formulate an arbitrage-free model of the term structure of interest rates. In this approach the short rate is driven by capital market returns. It is therefore essential the underlying market is modeled without using a bank account.

We found that surprisingly little is known about dynamic MV efficient strategies in continuous time if there is no short rate. Even in markets with deterministic or constant parameters, the generalization of single-period MV results to a dynamic continuous-time framework has been given only in the presence of a riskless asset. The absence of a short rate complicates the MV analysis. That is, in a deterministic framework, buy-and-hold investments in only two MV efficient strategies span all MV efficient strategies. In the presence of a short rate, a buy-and-hold investment in the bank account would be one of them.

Furthermore, Bekker and Bouwman (2009a) describe the value processes of stochastic discount factors as price processes in dual markets. Even when the primal markets have bank accounts, these dual markets do not have (dual versions) of bank accounts. Yet the dynamic MV frontier of Section 3 can be applied to derive dynamic generalizations of the Hansen-Jagannathan (1991) bounds.

Of course, the dynamic mean-variance results are relevant for portfolio theory as well, which holds in particular for cases without a short rate. Early work in portfolio theory was strongly influenced by the single-period Mean-Variance (MV) analysis of Markowitz (1952, 1959).[1] For example, MV analysis lead to the the subsequent development of the Capital Asset Pricing Model (CAPM) by Sharpe (1964), Lintner (1965a,b) and Mossin (1966). Recently, MV analysis has been generalized to a multiperiod context by Li and Ng (2000) and Zhou and Li (2000) among others.

Multiperiod portfolio selection was pioneered by Merton (1969, 1971) in a continuous-time expected utility framework. Merton (1971) solves the multiperiod consumption-

[1] An analytical derivation for the single period MV problem can be found in Merton (1972).

investment problem in continuous time by using dynamic programming. Over the past decades, this work has been generalized substantially.[2]

The dynamic MV portfolio selection problem, as considered in this paper, is closely related to the expected utility approach. It is well-known that optimizing expected quadratic utility yields a solution to the MV problem, see e.g. Xia (2005). However, quadratic utility generates a negative marginal utility for relatively large levels of wealth and thereby violates the classical assumption of strict monotonicity of the utility function. Therefore, the results in the expected utility literature are not directly applicable to the multiperiod MV problem. In fact, only recently the literature describes explicitly general results on MV frontiers.

In discrete time, and for a market with deterministic market parameters, Li and Ng (2000) solved the multiperiod MV problem. Their approach is characterized by solving an auxiliary optimization problem, using dynamic programming, to obtain the solution to the MV portfolio selection problem. Leippold *et al.* (2004) used an alternative, geometric approach to describe the solution.

In continuous time, the MV portfolio selection has almost exclusively been studied in complete markets. Zhou and Li (2000) follow an approach analogous to Li and Ng (2000) to solve the MV problem for a complete market with deterministic market parameters. Their work is subsequently generalized to markets with stochastic parameters by Lim and Zhou (2002), and Bielecki *et al.* (2005). An early example of MV portfolio selection in continuous time, using a martingale approach, is given by Bajeux-Besnainou and Portait (1998).

It is remarkable to observe that little work has been done on MV portfolio selection in an incomplete market continuous-time setting. Lim (2004) and Basak and Chabakauri (2008) do consider MV portfolio selection in an incomplete market with stochastic market parameters, but still assumes the presence of a short rate. We are unaware of results on the continuous-time generalization of the MV portfolio selection problem for markets with only risky assets.

A related body of literature investigates the MV hedging of unattainable claims. In MV hedging, an unattainable claim is hedged such that the expected quadratic hedging errors are minimized. This problem is closely related to MV portfolio selection, because the MV portfolio selection problem corresponds to MV hedging of a constant claim given a fixed initial investment (see e.g. Lim (2004)). The MV hedging problem is studied in discrete

[2]Important generalizations are given by Karatzas *et al.* (1986), Karatzas *et al.* (1987) and Cox and Huang (1989). Generalizations to the incomplete market case are given by He and Pearson (1991), Karatzas *et al.* (1991) and Cvitanić and Karatzas (1992). See Schachermayer (2002), Korn (1997) and Karatzas and Shreve (1998) for an overview.

time by Schweizer (1995), Bertsimas *et al.* (2001) and Černý (2004) and in continuous time by Duffie and Richardson (1991), Schäl (1994), Gouriéroux *et al.* (1998) and Bertsimas *et al.* (2001).[3]

Despite the fact that much work has been done on multiperiod MV analysis so far, a uniform treatment of MV portfolio selection for a general incomplete market in both discrete and continuous time seems to be missing. This paper tries to fill this gap by solving the dynamic MV problem for a general incomplete market in both discrete and continuous time. We consider an incomplete market where the joint dynamics of prices and a vector of state variables are assumed Markov. We use dynamic programming to obtain the MV solution in a uniform way that applies to markets with or without a riskless asset. Our solution is expressed as an explicit function of the three MV parameters describing the MV frontier for a particular horizon.[4] For these MV parameters, conditions are derived in the form of a recursive system of PDE's.

As a result we find, similar to Merton's (1973) $M+2$ mutual fund theorem, that all MV efficient strategies reduce to investments in a set of at most $M+2$ mutual funds. When the market parameters are deterministic, so that $M = 0$, there are two mutual funds which are instantaneously MV efficient. For this case, which can be considered the most straightforward generalization of the single-period Markowitz market, the paper provides explicit solutions. These solutions show that the two instantaneous MV efficient mutual funds can also be formulated as two uniformly MV efficient funds, which are dynamically MV efficient for any horizon. So, independent of the investment horizon, all MV efficient strategies invest buy-and-hold in only two dynamically MV efficient strategies. For constant parameters we show the existence of a unique, symmetrically distributed, trend stationary, uniformly MV efficient strategy.

The outline of this paper is as follows. Section 2 sets up the framework and discusses some general results on MV portfolio selection. Section 3 derives the solution to the dynamic MV portfolio selection problem in continuous time. The discrete-time solution is derived analogously in Appendix A.1. Specific results regarding a market with deterministic parameters are highlighted and a simple example with stochastic market parameters is illustrated in continuous time. Section 4 concludes.

[3] See Schweizer (2001) for an overview.

[4] Notice the multi-period frontier can also be considered as a single-period frontier with an infinite number of assets, given by self-financing strategies, as considered by Hansen and Richard (1987).

2 The Framework and Single-Period Results

2.1 The market

Consider a market with N assets and frictionless trading at fixed trading times given by the set $\mathcal{T}$. In particular, we discriminate between the discrete time case and continuous time. The probability structure and flow of information is described by the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathcal{T}}, P\right)$. The $N\times 1$-dimensional asset price process $\boldsymbol{S}_t$ is a positive process adapted to the filtration $\mathcal{F}_t$ and is affected by an $M \times 1$-dimensional adapted process of state variables $\boldsymbol{Z}_t$. The joint $(N+M)\times 1$ vector process $(\boldsymbol{S}_t', \boldsymbol{Z}_t')'$ is assumed Markov with finite second moment.[5]

In continuous time, that is $\mathcal{T} = [0, T]$, the dynamics of the asset prices and state variables are given by the following system of stochastic differential equations

$$\mathrm{d}\boldsymbol{S}_t = \mathrm{diag}(\boldsymbol{S}_t)\left\{\boldsymbol{\mu}_S(t, \boldsymbol{Z}_t)\,\mathrm{d}t + \boldsymbol{\sigma}_S(t, \boldsymbol{Z}_t)\,\mathrm{d}\boldsymbol{W}_t\right\},$$
$$\mathrm{d}\boldsymbol{Z}_t = \boldsymbol{\mu}_Z(t, \boldsymbol{Z}_t)\,\mathrm{d}t + \boldsymbol{\sigma}_Z(t, \boldsymbol{Z}_t)\,\mathrm{d}\boldsymbol{W}_t,$$

where $\boldsymbol{W}_t$ is L-dimensional standard Brownian motion adapted to $\mathcal{F}_t$ and the functions $\boldsymbol{\mu}_S$, $\boldsymbol{\mu}_Z$, $\boldsymbol{\sigma}_S$ and $\boldsymbol{\sigma}_Z$ are assumed sufficiently regular for the system to have a unique strong solution with a finite second moment.[6] Denote the instantaneous covariance matrices as $\boldsymbol{\Sigma}_S = \boldsymbol{\sigma}_S\boldsymbol{\sigma}_S{}' : N \times N$, $\boldsymbol{\Sigma}_Z = \boldsymbol{\sigma}_Z\boldsymbol{\sigma}_Z{}' : M \times M$, and $\boldsymbol{\Sigma}_{SZ} = \boldsymbol{\sigma}_S\boldsymbol{\sigma}_Z{}' : N \times M$.

Portfolios are described by an adapted N-dimensional weight process $\boldsymbol{\phi}_t$, where each component represents the holdings in the corresponding asset denoted in monetary units. The portfolio value is given by $V_{\boldsymbol{\phi},t} = \boldsymbol{\imath}'\boldsymbol{\phi}_t$, where $\boldsymbol{\imath}$ is a vector of ones. Attention is restricted to the set of admissible portfolios Φ that satisfy the following two conditions. First, admissible portfolios are self-financing, so that the value dynamics for an admissible portfolio $\boldsymbol{\phi}_t$ is given by

$$\mathrm{d}V_{\boldsymbol{\phi},t} = \boldsymbol{\phi}_t'\left\{\boldsymbol{\mu}_S(t, \boldsymbol{Z}_t)\,\mathrm{d}t + \boldsymbol{\sigma}_S(t, \boldsymbol{Z}_t)\,\mathrm{d}\boldsymbol{W}_t\right\}. \tag{1}$$

Secondly, they satisfy $E\int_0^T \boldsymbol{\phi}_t'\boldsymbol{\phi}_t\,\mathrm{d}t < \infty$, which ensures $\mathrm{E}\left(V_{\boldsymbol{\phi},t}^2\right) < \infty$, for $t \in [0, T]$.

[5] We use boldface for vectors and matrices.

[6] For conditions see Duffie (2001) and references therein.

2.2 The frontier

We denote portfolio returns by $R_{\boldsymbol{\phi}}(t;T) = V_{\boldsymbol{\phi},T}/V_{\boldsymbol{\phi},t}$ and their conditional moments by

$$\begin{aligned} m_{\boldsymbol{\phi}} &= m_{\boldsymbol{\phi}}(t,\boldsymbol{z};T) = \mathrm{E}\left(R_{\boldsymbol{\phi}}(t;T) \mid \boldsymbol{Z}_t = \boldsymbol{z}\right), \\ s^2_{\boldsymbol{\phi}} &= s^2_{\boldsymbol{\phi}}(t,\boldsymbol{z};T) = \mathrm{E}\left(R^2_{\boldsymbol{\phi}}(t;T) \mid \boldsymbol{Z}_t = \boldsymbol{z}\right), \\ v^2_{\boldsymbol{\phi}} &= s^2_{\boldsymbol{\phi}} - m^2_{\boldsymbol{\phi}}. \end{aligned}$$

The MV-frontier over a period $[t,T]$ can be considered a single-period frontier where assets are given by strategies. The frontier where the minimal value of either $s^2_{\boldsymbol{\phi}}$ or $v^2_{\boldsymbol{\phi}}$ is expressed as function of $m_{\boldsymbol{\phi}} = m$ is given by

$$\begin{aligned} s^2(m) &= \min_{\boldsymbol{\phi}} \left(s^2_{\boldsymbol{\phi}}(t,\boldsymbol{z};T) \mid m_{\boldsymbol{\phi}}(t,\boldsymbol{z};T) = m\right), \\ &= s^2_{\mathrm{LSR}} + \frac{(m - m_{\mathrm{LSR}})^2}{F^2} = m^2 + v^2_{\mathrm{GMV}} + \frac{(m - m_{\mathrm{GMV}})^2}{\Gamma^2}, \end{aligned}$$

where m_{LSR} and s^2_{LSR} are the first two moments of the *least square return* (LSR) portfolio that minimizes $s^2_{\boldsymbol{\phi}}$, and m_{GMV} and s^2_{GMV} are the first two moments of the *global minimum variance* (GMV) portfolio that minimizes $v^2_{\boldsymbol{\phi}}$. The scalars F and Γ do not depend on m and satisfy $\Gamma^2 = F^2/(1-F^2)$, and

$$m_{\mathrm{GMV}} = \frac{m_{\mathrm{LSR}}}{1-F^2}, \qquad v^2_{\mathrm{GMV}} = s^2_{\mathrm{LSR}} - \frac{m^2_{\mathrm{LSR}}}{1-F^2}. \tag{2}$$

2.3 MV efficient strategies and the value function

The MV portfolio selection problem is equivalent to the problem of MV hedging a constant claim C given a fixed initial investment (see e.g. Lim (2004)). This can be seen by considering the *value function* of this MV hedging problem, which is given by

$$\begin{aligned} J(t,\boldsymbol{z},x) &= \min_{\boldsymbol{\phi}} \mathrm{E}\{(V_{\boldsymbol{\phi},T} - C)^2 \mid \boldsymbol{Z}_t = \boldsymbol{z},\ V_{\boldsymbol{\phi},t} = x\} \qquad (3) \\ &= \min_{\boldsymbol{\phi}} \mathrm{E}\{(xR_{\boldsymbol{\phi}}(t;T) - C)^2 \mid \boldsymbol{Z}_t = \boldsymbol{z}\} \\ &= \min_{\boldsymbol{\phi}} \left\{x^2 s^2_{\boldsymbol{\phi}}(t,\boldsymbol{z};T) - 2x m_{\boldsymbol{\phi}}(t,\boldsymbol{z};T) C + C^2\right\}, \end{aligned}$$

Clearly the minimum is found for a MV efficient strategy. So, for $F > 0$ we have

$$J(t,\boldsymbol{z},x) = \min_{m} \left[x^2 \left\{s^2_{\mathrm{LSR}}(t,\boldsymbol{z};T) + \frac{\{m - m_{\mathrm{LSR}}(t,\boldsymbol{z};T)\}^2}{F^2(t,\boldsymbol{z};T)}\right\} - 2xmC + C^2\right],$$

and the minimum is found for $m = m_{\mathrm{LSR}}(t, \boldsymbol{z}; T) + F^2(t, \boldsymbol{z}; T)C/x$. Therefore, we have

$$J(t, \boldsymbol{z}, x) = \begin{pmatrix} x \\ -C \end{pmatrix}' \begin{pmatrix} s^2_{\mathrm{LSR}}(t, \boldsymbol{z}; T) & m_{\mathrm{LSR}}(t, \boldsymbol{z}; T) \\ m_{\mathrm{LSR}}(t, \boldsymbol{z}; T) & 1 - F^2(t, \boldsymbol{z}; T) \end{pmatrix} \begin{pmatrix} x \\ -C \end{pmatrix}. \tag{4}$$

Let he *minimum risk* (MR) strategy be defined as the strategy that minimizes the value function over the initial investment, where we assume $C \neq 0$,

$$\begin{aligned} Q(t, \boldsymbol{z}) &= \min_{x,\boldsymbol{\phi}} \mathrm{E}\{(V_{\boldsymbol{\phi},T} - C)^2 \mid \boldsymbol{Z}_t = \boldsymbol{z}, V_{\boldsymbol{\phi},t} = x\} \\ &= \min_{x,\boldsymbol{\phi}} \{x^2 s^2_{\boldsymbol{\phi}}(t, \boldsymbol{z}; T) - 2x m_{\boldsymbol{\phi}}(t, \boldsymbol{z}; T)C + C^2\} \\ &= \min_{\boldsymbol{\phi}} C^2 \left\{1 - \frac{m^2_{\boldsymbol{\phi}}(t, \boldsymbol{z}; T)}{s^2_{\boldsymbol{\phi}}(t, \boldsymbol{z}; T)}\right\} \\ &= C^2 \left\{1 - \frac{m^2_{\mathrm{MR}}(t, \boldsymbol{z}; T)}{s^2_{\mathrm{MR}}(t, \boldsymbol{z}; T)}\right\}, \end{aligned}$$

where x is found equal to $x = C m_{\mathrm{MR}}/s^2_{\mathrm{MR}}$. So MR also maximizes m/s or m/v. Alternatively, by minimizing (4), the solution is found for

$$\begin{aligned} x &= C m_{\mathrm{LSR}}/s^2_{\mathrm{LSR}}, \\ Q(t, \boldsymbol{z}) &= C^2 \left\{1 - F^2(t, \boldsymbol{z}; T) - \frac{m^2_{\mathrm{LSR}}(t, \boldsymbol{z}; T)}{s^2_{\mathrm{LSR}}(t, \boldsymbol{z}; T)}\right\}. \end{aligned}$$

Consequently, we find

$$\frac{m_{\mathrm{MR}}}{s^2_{\mathrm{MR}}} = \frac{m_{\mathrm{LSR}}}{s^2_{\mathrm{LSR}}}, \qquad F^2 = \frac{m^2_{\mathrm{MR}}}{s^2_{\mathrm{MR}}} - \frac{m^2_{\mathrm{LSR}}}{s^2_{\mathrm{LSR}}}, \qquad m_{\mathrm{MR}} = m_{\mathrm{LSR}} + \frac{s^2_{\mathrm{LSR}}}{m_{\mathrm{LSR}}} F^2.$$

As is well-known, two MV efficient returns span the MV frontier (if $F = 0$ a single return suffices). In particular the MV efficient portfolio that minimizes the value function (3) is given by

$$V_{\boldsymbol{\phi},T} = x R_{\boldsymbol{\phi}} = \frac{m_{\mathrm{MR}}}{s^2_{\mathrm{MR}}} C R_{\mathrm{MR}} + \left(x - \frac{m_{\mathrm{MR}}}{s^2_{\mathrm{MR}}} C\right) R_{\mathrm{LSR}}, \tag{5}$$

which follows from the optimality of R_{MR} and R_{LSR}.[7]

[7] That is, consider general portfolio values, with a starting value equal to x, given by $V_{\boldsymbol{\phi},T} + \varepsilon$, where $V_{\boldsymbol{\phi},T}$ equals (5) and ε is the value at time T of a zero-cost portfolio. The optimality of R_{MR} and R_{LSR} implies $\mathrm{E}\left\{\left(R_{\mathrm{MR}} - s^2_{\mathrm{MR}}/m_{\mathrm{MR}}\right) R\right\} = 0$, and $\mathrm{E}\left\{R_{\mathrm{LSR}}(R - R_{\mathrm{LSR}})\right\} = 0$, respectively, for arbitrary returns R. Consequently, $\mathrm{E}(R_{\mathrm{LSR}}\varepsilon) = 0$ and $\mathrm{E}(R_{\mathrm{MR}}\varepsilon) = \mathrm{E}(\varepsilon) s^2_{\mathrm{MR}}/m_{\mathrm{MR}}$, and $\mathrm{E}(V_{\boldsymbol{\phi},T} + \varepsilon - C)^2 = \mathrm{E}(V_{\boldsymbol{\phi},T} - C)^2 + \mathrm{E}(\varepsilon^2)$. So (3) is minimized by (5).

2.4 The instantaneous frontier

If the mean and variance of the instantaneous return of the portfolio are given by

$$m_{\boldsymbol{\phi}}(t, \boldsymbol{z}, t + \mathrm{d}t) = 1 + \mu_{\boldsymbol{\phi}}(t, \boldsymbol{z})\,\mathrm{d}t,$$
$$v^2_{\boldsymbol{\phi}}(t, \boldsymbol{z}, t + \mathrm{d}t) = \sigma^2_{\boldsymbol{\phi}}(t, \boldsymbol{z})\,\mathrm{d}t,$$

respectively, then the instantaneous MV-frontier can be represented as

$$\min_{\boldsymbol{\phi}} \left(\sigma^2_{\boldsymbol{\phi}}(t, \boldsymbol{z}) \mid \mu_{\boldsymbol{\phi}}(t, \boldsymbol{z}) = \mu\right) = \alpha^2(t, \boldsymbol{z}) + \frac{(\mu - \beta(t, \boldsymbol{z}))^2}{\gamma^2(t, \boldsymbol{z})},$$

with instantaneous MV parameters α, β and γ.

If there is no riskless asset, we assume $\boldsymbol{\Sigma}_S$ is nonsingular. In case there is a bank account, we assume nonsingularity of the $(N-1) \times (N-1)$-matrix $\boldsymbol{\Sigma}_{22}(t, \boldsymbol{z})$, where

$$\boldsymbol{\Sigma}_S(t, \boldsymbol{z}) = \begin{pmatrix} 0 & \boldsymbol{0}' \\ \boldsymbol{0} & \boldsymbol{\Sigma}_{22}(t, \boldsymbol{z}) \end{pmatrix}.$$

To cover both cases we assume the existence of the following inverse for all t and $\boldsymbol{z}$

$$\begin{pmatrix} \boldsymbol{H} & \boldsymbol{h} \\ \boldsymbol{h}' & h_{22} \end{pmatrix} = \begin{pmatrix} \boldsymbol{H}(t, \boldsymbol{z}) & \boldsymbol{h}(t, \boldsymbol{z}) \\ \boldsymbol{h}(t, \boldsymbol{z})' & h_{22}(t, \boldsymbol{z}) \end{pmatrix} = \begin{pmatrix} \boldsymbol{\Sigma}_S(t, \boldsymbol{z}) & \boldsymbol{\imath} \\ \boldsymbol{\imath}' & 0 \end{pmatrix}^{-1}. \tag{6}$$

For the two cases, i.e. when all assets are risky and when there is a riskless asset, we find

$$\boldsymbol{h} = \frac{\boldsymbol{\Sigma}_S^{-1}\boldsymbol{\imath}}{\boldsymbol{\imath}'\boldsymbol{\Sigma}_S^{-1}\boldsymbol{\imath}}, \qquad \boldsymbol{H} = \boldsymbol{\Sigma}_S^{-1} - \frac{\boldsymbol{\Sigma}_S^{-1}\boldsymbol{\imath}\boldsymbol{\imath}'\boldsymbol{\Sigma}_S^{-1}}{\boldsymbol{\imath}'\boldsymbol{\Sigma}_S^{-1}\boldsymbol{\imath}}, \qquad \left(h_{22} = \frac{1}{\boldsymbol{\imath}'\boldsymbol{\Sigma}_S^{-1}\boldsymbol{\imath}}\right),$$

and

$$\boldsymbol{h} = \begin{pmatrix} 1 \\ \boldsymbol{0} \end{pmatrix}, \qquad \boldsymbol{H} = \begin{pmatrix} \boldsymbol{\imath}'\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\imath} & -\boldsymbol{\imath}'\boldsymbol{\Sigma}_{22}^{-1} \\ -\boldsymbol{\Sigma}_{22}^{-1}\boldsymbol{\imath} & \boldsymbol{\Sigma}_{22}^{-1} \end{pmatrix}, \qquad (h_{22} = 0),$$

respectively. Notice that $\boldsymbol{h}'\boldsymbol{\imath} = 1$, $\boldsymbol{H}\boldsymbol{\imath} = \boldsymbol{0}$, $\boldsymbol{H}\boldsymbol{\Sigma}_S\boldsymbol{h} = \boldsymbol{0}$ and $\boldsymbol{H}\boldsymbol{\Sigma}_S\boldsymbol{H} = \boldsymbol{H}$.

Whether or not there is a riskless asset, the instantaneous MV-parameters are now given by

$$\alpha^2 = \alpha^2(t, \boldsymbol{z}) = \boldsymbol{h}'\boldsymbol{\Sigma}_S\boldsymbol{h}, \tag{7}$$

$$\beta = \beta(t, \boldsymbol{z}) = \boldsymbol{h}'\boldsymbol{\mu}_S, \tag{8}$$

$$\gamma^2 = \gamma^2(t, \boldsymbol{z}) = \boldsymbol{\mu}_S'\boldsymbol{H}\boldsymbol{\mu}_S. \tag{9}$$

The instantaneous MV efficient MR, GMV and LSR portfolios, with starting value 1, are given by

$$\boldsymbol{\phi}_{\text{MR}}(t, \boldsymbol{z}; t + \mathrm{d}t) = \boldsymbol{\phi}_{\text{GMV}}(t, \boldsymbol{z}; t + \mathrm{d}t) = \boldsymbol{h}, \tag{10}$$

$$\boldsymbol{\phi}_{\text{LSR}}(t, \boldsymbol{z}; t + \mathrm{d}t) = \boldsymbol{h} - \boldsymbol{H}\boldsymbol{\mu}_S. \tag{11}$$

3 Dynamic MV analysis in continuous time

To obtain the solution to the MV portfolio selection problem, we solve the following MV hedging problem

$$\min_{\boldsymbol{\phi}} \mathrm{E}\left\{(V_{\boldsymbol{\phi},T} - C)^2 \mid \boldsymbol{Z}_0 = \boldsymbol{z}_0,\ V_{\boldsymbol{\phi},0} = 1\right\}, \tag{12}$$

where C is a nonrandom scalar. We use the Bellman principle of optimality to obtain the solution to (12). In continuous time, the Bellman principle condenses to an optimality condition in the form of the well-known Hamilton-Jacobi-Bellman (HJB) equation. See Fleming and Rishel (1975), Øksendal (2003), Bjork (2004) or Chang (2004) for a discussion of dynamic programming in continuous time and the use of HJB equations.

The value function is given by (4) and its derivatives with respect to vectors $\boldsymbol{a}$ and $\boldsymbol{b}$, say, are denoted as

$$\boldsymbol{J}_{\boldsymbol{a}} = \frac{\partial J}{\partial \boldsymbol{a}}, \qquad \text{and} \qquad \boldsymbol{J}_{\boldsymbol{a}\boldsymbol{b}'} = \frac{\partial^2 J}{\partial \boldsymbol{a}\partial \boldsymbol{b}'}.$$

Let

$$A(t, \boldsymbol{z}, x) = J_t + \boldsymbol{\mu}_Z'\boldsymbol{J}_{\boldsymbol{z}} + \mathrm{tr}\left\{\boldsymbol{\Sigma}_Z\boldsymbol{J}_{\boldsymbol{z}\boldsymbol{z}'}\right\}/2,$$

$$B(t, \boldsymbol{z}, x) = \min_{\boldsymbol{\phi}_t}\left[\boldsymbol{\phi}_t'\left\{\boldsymbol{\mu}_S J_x + \boldsymbol{\Sigma}_{SZ}\boldsymbol{J}_{\boldsymbol{z}x}\right\} + \boldsymbol{\phi}_t'\boldsymbol{\Sigma}_S\boldsymbol{\phi}_t J_{xx}/2 \mid \boldsymbol{\imath}'\boldsymbol{\phi}_t = x\right],$$

then for all $x \in \mathbb{R}$, $\boldsymbol{z} \in \mathbb{R}^M$ and $t \in (0, T]$, the value function satisfies the following HJB equation:

$$A(t,\boldsymbol{z},x) + B(t,\boldsymbol{z},x) = 0, \tag{13}$$
$$J(T,\boldsymbol{z},x) = (x-C)^2. \tag{14}$$

Theorem 1. *The value function (3) is of the form (4) and the dynamic MV parameters* s^2_{LSR}, m_{LSR} *and* F^2: $[0,T]\times\mathbb{R}^M \mapsto \mathbb{R}$ *satisfy (with the arguments suppressed)*

$$\begin{aligned}\frac{\partial s^2_{LSR}}{\partial t} &+ \boldsymbol{\mu}'_{\mathrm{z}}\frac{\partial s^2_{LSR}}{\partial \boldsymbol{z}} + \frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{z}}\frac{\partial^2 s^2_{LSR}}{\partial \boldsymbol{z}\partial \boldsymbol{z}'}\right) + \\ &s^2_{LSR}\left\{\boldsymbol{h}'\boldsymbol{\Sigma}_{\mathrm{S}}\boldsymbol{h} + 2\boldsymbol{h}'\left(\boldsymbol{\mu}_{\mathrm{S}} + \boldsymbol{\Sigma}_{\mathrm{SZ}}\frac{\partial \log(s^2_{LSR})}{\partial \boldsymbol{z}}\right) - \right.\\ &\left.\left(\boldsymbol{\mu}_{\mathrm{S}} + \boldsymbol{\Sigma}_{\mathrm{SZ}}\frac{\partial \log(s^2_{LSR})}{\partial \boldsymbol{z}}\right)'\boldsymbol{H}\left(\boldsymbol{\mu}_{\mathrm{S}} + \boldsymbol{\Sigma}_{\mathrm{SZ}}\frac{\partial \log(s^2_{LSR})}{\partial \boldsymbol{z}}\right)\right\} = 0, \end{aligned}\tag{15}$$

$$\begin{aligned}\frac{\partial m_{LSR}}{\partial t} &+ \boldsymbol{\mu}'_{\mathrm{z}}\frac{\partial m_{LSR}}{\partial \boldsymbol{z}} + \frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{z}}\frac{\partial^2 m_{LSR}}{\partial \boldsymbol{z}\partial \boldsymbol{z}'}\right) + \\ &m_{LSR}\left\{\boldsymbol{h}'\left(\boldsymbol{\mu}_{\mathrm{S}} + \boldsymbol{\Sigma}_{\mathrm{SZ}}\frac{\partial \log(m_{LSR})}{\partial \boldsymbol{z}}\right) - \right.\\ &\left.\left(\boldsymbol{\mu}_{\mathrm{S}} + \boldsymbol{\Sigma}_{\mathrm{SZ}}\frac{\partial \log(s^2_{LSR})}{\partial \boldsymbol{z}}\right)'\boldsymbol{H}\left(\boldsymbol{\mu}_{\mathrm{S}} + \boldsymbol{\Sigma}_{\mathrm{SZ}}\frac{\partial \log(m_{LSR})}{\partial \boldsymbol{z}}\right)\right\} = 0, \end{aligned}\tag{16}$$

$$\begin{aligned}\frac{\partial F^2}{\partial t} &+ \boldsymbol{\mu}'_{\mathrm{z}}\frac{\partial F^2}{\partial \boldsymbol{z}} + \frac{1}{2}\,\mathrm{tr}\left(\boldsymbol{\Sigma}_{\mathrm{z}}\frac{\partial^2 F^2}{\partial \boldsymbol{z}\partial \boldsymbol{z}'}\right) + \\ &\frac{m^2_{LSR}}{s^2_{LSR}}\left(\boldsymbol{\mu}_{\mathrm{S}} + \boldsymbol{\Sigma}_{\mathrm{SZ}}\frac{\partial \log(m_{LSR})}{\partial \boldsymbol{z}}\right)'\boldsymbol{H}\left(\boldsymbol{\mu}_{\mathrm{S}} + \boldsymbol{\Sigma}_{\mathrm{SZ}}\frac{\partial \log(m_{LSR})}{\partial \boldsymbol{z}}\right) = 0, \end{aligned}\tag{17}$$

with boundary conditions $m_{LSR}(T,\boldsymbol{z};T) = s^2_{LSR}(T,\boldsymbol{z};T) = 1 - F^2(T,\boldsymbol{z};T) = 1$.

The portfolio weights $\boldsymbol{\phi}_t$ *for the strategy* (1) *that minimizes (12) are given by*

$$\begin{aligned}\boldsymbol{\phi}_t &= \left\{\boldsymbol{h} + \boldsymbol{H}\boldsymbol{\Sigma}_{\mathrm{SZ}}\frac{\partial \log(m_{LSR}/s^2_{LSR})}{\partial \boldsymbol{z}}\right\}C\frac{m_{LSR}}{s^2_{LSR}} + \\ &\quad\left(\boldsymbol{h} - \boldsymbol{H}\boldsymbol{\mu}_{\mathrm{S}} - \boldsymbol{H}\boldsymbol{\Sigma}_{\mathrm{SZ}}\frac{\partial \log s^2_{LSR}}{\partial \boldsymbol{z}}\right)\left(V_{\boldsymbol{\phi},t} - C\frac{m_{LSR}}{s^2_{LSR}}\right). \end{aligned}\tag{18}$$

The proof is given in Appendix A.2. Notice, in a complete market a riskless bond amounts to an investment in a term-T global minimum variance strategy with variance $v^2_{\mathrm{GMV}} = 0$. So, using (2), a term-T bond price that pays $B_T = 1$ at time T has value $B_t = m^{-1}_{\mathrm{GMV}} = m_{\mathrm{LSR}}/s^2_{\mathrm{LSR}}$. Consequently, the term $Cm_{\mathrm{LSR}}/s^2_{\mathrm{LSR}}$ can be interpreted as the present value of the claim that has value C at time T. It is the difference between the portfolio value and the present value of the claim that drives the mean reversion of MV efficient processes.

The dynamic MV portfolio selection problem in discrete time can be solved analogously and the solution is given in Appendix A.1. For a discussion of the theorem we will first focus on the deterministic case, where there are no state variables. Subsequently, the effect of the state variables will be considered.

3.1 The deterministic case

In the deterministic case, where there are no state variables, we can use the definitions of α, β, and γ in (7), (8), and (9), to reformulate the conditions (15), (16), and (17) as

$$\frac{\mathrm{d}s^2_{\mathrm{LSR}}}{\mathrm{d}t} = -s^2_{\mathrm{LSR}}(2\beta + \alpha^2 - \gamma^2), \qquad \frac{\mathrm{d}m_{\mathrm{LSR}}}{\mathrm{d}t} = -m_{\mathrm{LSR}}(\beta - \gamma^2), \qquad \frac{\mathrm{d}F^2}{\mathrm{d}t} = -\frac{m^2_{\mathrm{LSR}}}{s^2_{\mathrm{LSR}}}\gamma^2,$$

respectively. In that case we find

$$s^2_{\mathrm{LSR}} = e^{\int_t^T \{2\beta(s)+\alpha^2(s)-\gamma^2(s)\}\,\mathrm{d}s}, \tag{19}$$

$$m_{\mathrm{LSR}} = e^{\int_t^T \{\beta(s)-\gamma^2(s)\}\,\mathrm{d}s}, \tag{20}$$

$$F^2 = \int_t^T \gamma^2(s) e^{-\int_s^T \{\alpha^2(u)+\gamma^2(u)\}\,\mathrm{d}u}\,\mathrm{d}s, \tag{21}$$

and the portfolio weights are given by

$$\boldsymbol{\phi}_t = \boldsymbol{h} C e^{-\int_t^T \{\beta(s)+\alpha^2(s)\}\,\mathrm{d}s} + (\boldsymbol{h} - \boldsymbol{H}\boldsymbol{\mu}_S)\left(V_{\boldsymbol{\phi},t} - C e^{-\int_t^T \{\beta(s)+\alpha^2(s)\}\,\mathrm{d}s}\right). \tag{22}$$

Notice the weights $\boldsymbol{h}$ and $\boldsymbol{h} - \boldsymbol{H}\boldsymbol{\mu}_S$ represent the instantaneous MR (or GMV) and LSR portfolios, as in (10) and (11), respectively. The LSR and MR strategies starting at time $t = 0$, found for $C = 0$ and $C = s^2_{\mathrm{LSR}}(0;T)/m_{\mathrm{LSR}}(0;T)$ respectively, are given by

$$\boldsymbol{\phi}_{\mathrm{LSR},t} = (\boldsymbol{h} - \boldsymbol{H}\boldsymbol{\mu}_S) V_{\mathrm{LSR},t}, \tag{23}$$

$$\boldsymbol{\phi}_{\mathrm{MR},t} = \boldsymbol{h} e^{\int_0^t \{\beta(s)+\alpha^2(s)\}\,\mathrm{d}s} + (\boldsymbol{h} - \boldsymbol{H}\boldsymbol{\mu}_S)\left(V_{\mathrm{MR},t} - e^{\int_0^t \{\beta(s)+\alpha^2(s)\}\,\mathrm{d}s}\right). \tag{24}$$

The MV efficient strategies (22) are affine combinations,

$$\boldsymbol{\phi}_t = K\boldsymbol{\phi}_{\mathrm{MR},t} + (1-K)\boldsymbol{\phi}_{\mathrm{LSR},t}, \qquad K = C e^{-\int_0^T \{\beta(s)+\alpha^2(s)\}\,\mathrm{d}s}.$$

Notice for fixed K these term-T MV efficient strategies do not depend on T. That is, the LSR and MR strategies span all frontiers of varying terms T. So, in the continuous-time deterministic case we find that MV efficient strategies are *uniformly MV efficient*.

Notice that the LSR strategy amounts to investing continuously in the instantaneous

LSR portfolio, with relative weights given by $\boldsymbol{h}-\boldsymbol{H}\boldsymbol{\mu}_S$. If there is a riskless asset, i.e. when $\alpha^2 = 0$ for all t, then also the MR strategy amounts to repeating the same instantaneous steps. That is, in that case MR amounts to continuously investing in the bank account with (deterministic) short rate equal to β, and $V_{\text{MR},t} = \exp \int_0^t \{\beta(s)\} \,\mathrm{d}s$. For this case, uniform MV efficiency (or strong separability) has been described by Bajeux-Besnainou and Portait (1998).

When there is no risk-free asset the MR strategy is more complicated. It starts out by investing in the instantaneous GMV portfolio, which equals the instantaneous MR, with weights $\boldsymbol{h}$. At time t a deterministic amount $\exp \int_0^t \{\beta(s) + \alpha^2(s)\} \,\mathrm{d}s$ is invested in the instantaneous GMV, while the rest of the portfolio value is invested in the instantaneous LSR. Still we find uniform MV efficiency. Consequently, Merton's (1972, 1973) two mutual fund theorems can be formulated in terms of two funds that are not only instantaneously MV efficient, as it was described by Merton, but which are MV efficient for *all* terms.

The LSR and MR strategies are not only uniformly MV efficient, they are also uniformly LSR and MR strategies. By contrast, the term-T GMV strategy is uniformly MV efficient, but it is GMV only for term T. That is, the GMV strategy for term T is found for

$$K_{\text{GMV}} = \frac{m_{\text{GMV}}(0;T) - m_{\text{LSR}}(0:T)}{m_{\text{MR}}(0;T) - m_{\text{LSR}}(0;T)} = \frac{m^2_{\text{LSR}}(0;T)}{s^2_{\text{LSR}}(0;T)\{1 - F^2(0;T)\}},$$

which varies with T.

3.1.1 Constant parameters

In particular, if the parameters α^2, β, and γ^2 are constant, we find a remarkable result. The uniformly MV efficient processes can be described by

$$\mathrm{d}V_t = V_t \left(\beta \,\mathrm{d}t + \alpha \,\mathrm{d}W_t^{(1)}\right) + \left(Ke^{(\beta+\alpha^2)t} - V_t\right) \left(\gamma^2 \,\mathrm{d}t + \gamma \,\mathrm{d}W_t^{(2)}\right), \tag{25}$$

where $W_t^{(1)} = \boldsymbol{h}'\boldsymbol{\sigma}_S \boldsymbol{W}_t/\alpha$ and $W_t^{(2)} = \boldsymbol{\mu}_S'\boldsymbol{H}\boldsymbol{\sigma}_S\boldsymbol{W}_t/\gamma$ are the Brownian motions that drive the MV-efficient strategies. For $K = 0$ we find the LSR strategy with growth rate $\beta - \gamma^2$, but $e^{-(\beta-\gamma^2)t}V_{\text{LSR},t}$ is not stationary due to its increasing variance. For $K \neq 0$ we find the other MV efficient strategies, which all converge to investments in a unique trend stationary process with growth rate $\beta + \alpha^2$. That is, applying Ito's Lemma to (25) shows the discounted processes $V_t^* = e^{-(\beta+\alpha^2)t}V_t/K$ all satisfy

$$\mathrm{d}V_t^* = V_t^* \left(-\alpha^2 \,\mathrm{d}t + \alpha \,\mathrm{d}W_t^{(1)}\right) + (1 - V_t^*) \left(\gamma^2 \,\mathrm{d}t + \gamma \,\mathrm{d}W_t^{(2)}\right). \tag{26}$$

Only the starting values $V_0^* = 1/K$ differ. All processes converge to a stationary process with equilibrium value $\overline{V}^* = \gamma^2/(\alpha^2+\gamma^2)$ and unconditional standard deviation equal to $\alpha\gamma/(\alpha^2+\gamma^2)$.

In fact, the unconditional distribution of the stationary process is symmetric. That is, consider a value $V_t^* = \overline{V}^* + \Delta$, and using the notation $\mathrm{d}V_t^* = \mu\,\mathrm{d}t + \sigma\,\mathrm{d}W_t$, we find $\mu = -\Delta(\alpha^2+\gamma^2)$ and $\sigma^2 = \alpha^2\gamma^2/(\alpha^2+\gamma^2) + \Delta^2(\alpha^2+\gamma^2)$. We find that opposite values Δ and $-\Delta$ produce opposite μ's but the same σ^2. Due to this symmetry the unconditional distribution of V_t^* must be symmetric.

Finally, this unique uniformly MV efficient trend stationary (UMVETS) strategy, which has a symmetric distribution, has another interesting property. When starting in equilibrium, $K = 1+\alpha^2/\gamma^2$, it forms with LSR a unique pair of uniformly MV efficient strategies whose values are uncorrelated for all t.[8]

For the constant parameters case we find the following moments

$$m_{\mathrm{LSR}} = e^{(\beta-\gamma^2)(T-t)}, \qquad v^2_{\mathrm{LSR}} = e^{(2\beta+\alpha^2-\gamma^2)(T-t)} - e^{2(\beta-\gamma^2)(T-t)},$$

$$m_{\mathrm{GMV}} = \frac{m_{\mathrm{LSR}}(\alpha^2+\gamma^2)}{\alpha^2+\gamma^2 e^{-(\alpha^2+\gamma^2)(T-t)}}, \qquad v^2_{\mathrm{GMV}} = \frac{\alpha^2 v^2_{\mathrm{LSR}}}{\alpha^2+\gamma^2 e^{-(\alpha^2+\gamma^2)(T-t)}},$$

$$m_{\mathrm{MR}} = \frac{m_{\mathrm{LSR}}\{\alpha^2+\gamma^2 e^{(\alpha^2+\gamma^2)(T-t)}\}}{\alpha^2+\gamma^2}, \qquad v^2_{\mathrm{MR}} = \frac{\alpha^2 v^2_{\mathrm{LSR}}\{\alpha^2+\gamma^2 e^{(\alpha^2+\gamma^2)(T-t)}\}}{(\alpha^2+\gamma^2)^2},$$

and frontier parameters

$$F^2 = \frac{\gamma^2\{1-e^{-(\alpha^2+\gamma^2)(T-t)}\}}{\alpha^2+\gamma^2}, \qquad \Gamma^2 = \frac{\gamma^2\{1-e^{(\alpha^2+\gamma^2)(T-t)}\}}{\alpha^2+\gamma^2 e^{-(\alpha^2+\gamma^2)(T-t)}}.$$

Thus we find $K_{\mathrm{GMV}} \to 0$ if $T \to \infty$. Furthermore, if $T \to \infty$, $v^2_{\mathrm{GMV}}/m^2_{\mathrm{GMV}} \to \infty$ and $v^2_{\mathrm{LSR}}/m^2_{\mathrm{LSR}} \to \infty$, whereas $v^2_{\mathrm{MR}}/m^2_{\mathrm{MR}} \to \alpha^2/\gamma^2$.

Bekker and Bouwman (2009b) model the term structure of interest rates. They use an underlying market of capital market returns, without a bank account, that drives the short rate. In this approach the growth rate of the unique UMVETS strategy, $\beta+\alpha^2$, serves as the growth rate of the market portfolio of the underlying market. Using Sharpe ratio optimality of the market portfolio, a stochastic short rate is induced by the underlying capital market returns. The growth rate of the LSR portfolio, $\beta-\gamma^2$, or the growth rate of the term-T GMV portfolio where $T \to \infty$, serves as the growth rate of the long bond with maturity $T \to \infty$.

[8]The computations are somewhat tedious.

3.2 The effect of state variables

In the presence of state variables the portfolio weights $\boldsymbol{\phi}_t$ in (18) are located in the range space of $(\boldsymbol{h}, \boldsymbol{H}\boldsymbol{\mu}_S, \boldsymbol{H}\boldsymbol{\Sigma}_{SZ})$. This indicates $M+2$ mutual funds, which is in agreement with Merton's (1973) $M+2$ mutual fund theorem. In general it will not be the case that two strategies span all frontiers. For example, the instantaneous MR strategy invests at $t=0$ in $\boldsymbol{h}$, whereas the term-T MR strategy invests at time $t=0$ in $\boldsymbol{h} + \boldsymbol{H}\boldsymbol{\Sigma}_{SZ}\frac{\partial \log(m_{\text{LSR}}/s^2_{\text{LSR}})}{\partial \boldsymbol{z}}$, which may be different from $\boldsymbol{h}$.

An interesting special case is given by the situation where the state variables are instantaneously uncorrelated with the asset returns, i.e. $\boldsymbol{\Sigma}_{SZ} = \mathbf{O}$. In that case the portfolio weights (18) are simply given by

$$\boldsymbol{\phi}_t = \boldsymbol{h} C \frac{m_{\text{LSR}}}{s^2_{\text{LSR}}} + (\boldsymbol{h} - \boldsymbol{H}\boldsymbol{\mu}_S)\left(V_{\boldsymbol{\phi},t} - C\frac{m_{\text{LSR}}}{s^2_{\text{LSR}}}\right),$$

just as in the deterministic case. Again, the LSR strategy is given by $\boldsymbol{\phi}_{\text{LSR},t} = (\boldsymbol{h} - \boldsymbol{H}\boldsymbol{\mu}_S)V_{\text{LSR},t}$, which is uniformly MV efficient. However, the ratio $m_{\text{LSR}}/s^2_{\text{LSR}}$ may be stochastic and, as a result, a MR strategy may vary with T. So there need not be two uniformly MV efficient strategies.

Furthermore, by applying the Feynman-Kač theorem (Karatzas and Shreve, 1991) to (15), (16), and (17) with $\boldsymbol{\Sigma}_{SZ} = \mathbf{O}$, the MV parameters admit the following simple representation:

$$\begin{aligned}
s^2_{\text{LSR}} &= \mathrm{E}_t\left(e^{\int_t^T \{2\beta(s,\boldsymbol{Z}_s)+\alpha^2(s,\boldsymbol{Z}_s)-\gamma^2(s,\boldsymbol{Z}_s)\}\,\mathrm{d}s} \mid \boldsymbol{Z}_t = \boldsymbol{z}\right),\\
m_{\text{LSR}} &= \mathrm{E}_t\left(e^{\int_t^T \{\beta(s,\boldsymbol{Z}_s)-\gamma^2(s,\boldsymbol{Z}_s)\}\,\mathrm{d}s} \mid \boldsymbol{Z}_t = \boldsymbol{z}\right),\\
F^2 &= \mathrm{E}_t\left(\int_t^T \gamma^2(s,\boldsymbol{Z}_s)e^{-\int_s^T \{\alpha^2(u,\boldsymbol{Z}_u)+\gamma^2(u,\boldsymbol{Z}_u)\}\,\mathrm{d}u}\,\mathrm{d}s \mid \boldsymbol{Z}_t = \boldsymbol{z}\right).
\end{aligned}$$

Example 1. *Consider the following market of two assets with instantaneous returns equal to the instantaneous GMV and LSR returns, respectively, and a single mean reverting state variable affecting the level of returns:*

$$\boldsymbol{\mu}_{\text{S}}(t,z) = \begin{pmatrix} z \\ z-\gamma^2 \end{pmatrix}, \qquad \boldsymbol{\sigma}_{\text{S}} = \begin{pmatrix} \alpha & 0 & 0 \\ \alpha & \gamma & 0 \end{pmatrix},$$

$$\mu_{\text{Z}}(t,z) = \kappa(\beta_o - z), \qquad \boldsymbol{\sigma}_{\text{Z}} = (0,0,\sigma_o).$$

The instantaneous parameters α and γ satisfy (7) and (9), respectively, and $\beta = z$. The instantaneous MV efficient portfolio weights are given by $\boldsymbol{h} = (1,0)'$ and $\boldsymbol{h} - \boldsymbol{H}\boldsymbol{\mu}_{\text{S}} = (0,1)'$.

If $\alpha = 0$, the first asset is risk-free, with a stochastic short rate equal to z, otherwise both assets are risky.

The MV parameters s^2_{LSR}, m_{LSR} and F^2 satisfy (15), (16) and (17), with $\boldsymbol{\Sigma}_{\mathrm{sz}} = \mathbf{O}$, and the boundary conditions. The solution is derived in Appendix A.4:

$$s^2_{LSR} = \exp\left[\left(2\beta_o + \alpha^2 - \gamma^2 + \frac{2\sigma_o^2}{\kappa^2}\right)(T-t) - \left(\frac{2(\beta_0 - z)}{\kappa} + \frac{4\sigma_o^2}{\kappa^3}\right)\left\{1 - e^{-\kappa(T-t)}\right\} \right. \tag{27}$$
$$\left. + \frac{\sigma_o^2}{\kappa^3}\left\{1 - e^{-2\kappa(T-t)}\right\}\right],$$

$$m_{LSR} = \exp\left[\left(\beta_o - \gamma^2 + \frac{\sigma_o^2}{2\kappa^2}\right)(T-t) - \left(\frac{\beta_0 - z}{\kappa} + \frac{\sigma_o^2}{\kappa^3}\right)\left\{1 - e^{-\kappa(T-t)}\right\} \right. \tag{28}$$
$$\left. + \frac{\sigma_o^2}{4\kappa^3}\left\{1 - e^{-2\kappa(T-t)}\right\}\right],$$

$$F^2 = \gamma^2 \int_t^T \frac{m^2_{LSR}(u, z; T)}{s^2_{LSR}(u, z; T)}\,\mathrm{d}u. \tag{29}$$

We find m^2_{LSR}/s^2_{LSR} is deterministic, since it does not depend on z. As a result, F^2 is also deterministic. A MV efficient strategy over the period $[0, T]$ is given by

$$\boldsymbol{\phi}_t = \boldsymbol{h} C \frac{m_{LSR}}{s^2_{LSR}} + (\boldsymbol{h} - \boldsymbol{H}\boldsymbol{\mu}_{\mathrm{S}})\left(V_{\boldsymbol{\phi},t} - C\frac{m_{LSR}}{s^2_{LSR}}\right).$$

Comparing these portfolio weights $\boldsymbol{\phi}_t$ with the portfolio weights $\boldsymbol{\phi}^*_t$ that are optimal over the subperiod $[0, T^*]$ with $0 < T^* < T$ implies that $\boldsymbol{\phi}_t$ can only be MV efficient over $[0, T^*]$ if there exists a $C^* \in \mathbb{R}$ such that

$$C\frac{m_{LSR}(t, z; T)}{s^2_{LSR}(t, z; T)} = C^* \frac{m_{LSR}(t, z; T^*)}{s^2_{LSR}(t, z; T^*)}.$$

However, $\frac{m_{LSR}(t,z;T)}{s^2_{LSR}(t,z;T)}$ depends on $\boldsymbol{z}$ for $t < T$, while it does not for $t = T$. For $C \neq 0$, we therefore cannot find a $C^* \in \mathbb{R}$ that satisfies the above condition and hence $\boldsymbol{\phi}_t$ is not uniformly efficient. For $C = 0$ we obtain the LSR strategy, which is evidently uniformly LSR and thus uniformly MV efficient.

Finally, observe that for $\alpha = 0$ the market has a Vasicek (1977) short rate process.

4 Conclusion

Motivated by yield curve modeling, this paper solves the dynamic MV portfolio selection problem in both discrete and continuous time by using dynamic programming. The solution is derived for a general incomplete market that nests a complete market as well as an incomplete market with or without a riskless asset. State variables are introduced to incorporate stochastic market parameters. The joint process of asset prices and state variables is assumed Markov.

It is observed that the dynamic MV portfolio selection problem is equivalent to MV hedging of a constant claim C given a fixed initial investment. The value function of the MV hedging problem is expressed as a quadratic form that is driven by three MV parameters describing the MV frontier. In continuous time, we obtain a recursive system of PDE's in the three MV parameters by solving the HJB equation. The optimal portfolio weights of a MV efficient strategy are expressed as a function of the MV parameters.

Explicit solutions to the MV problem are obtained for a market with deterministic parameters. All MV efficient strategies in this market are shown to be *uniformly* MV efficient, i.e. MV efficient on any term T. Consequently, a strong version of the Mutual Fund theorem holds, stating that all MV investors with arbitrary investment horizons invest buy-and-hold in two uniform MV efficient mutual funds.

The market with constant parameters provides the most straightforward generalization of one-period MV analysis to a dynamic continuous-time setting. In this market there exists a unique, symmetrically distributed, trend stationary, uniformly MV efficient strategy.

Appendix

A.1 Dynamic Mean-Variance analysis in discrete time

A.1.1 The market

In discrete time, the set of trading times is given by the set $\mathcal{T} = \{t_0, \ldots, t_n\}$, with $0 = t_0 < \ldots < t_n = T$. Define vectors of single-period gross returns $\boldsymbol{R}_{t_i} = \operatorname{diag}(\boldsymbol{S}_{t_{i-1}})^{-1}\boldsymbol{S}_{t_i}$, $i = 1, \ldots, n$, then the joint process $\left(\boldsymbol{R}'_{t_i}, \boldsymbol{Z}'_{t_i}\right)'$ is Markov as well. We assume that, conditional on $\boldsymbol{Z}_{t_i}$, both $\boldsymbol{R}_{t_{i+1}}$ and $\boldsymbol{Z}_{t_{i+1}}$ are independent of $\boldsymbol{R}_{t_i}$. The conditional moments of the single-period gross returns are denoted, for $i = 0, \ldots, n-1$, by[9]

$$\boldsymbol{m}_{S,i} = \boldsymbol{m}_S(t_i, \boldsymbol{z}; t_{i+1}) = \mathrm{E}(\boldsymbol{R}_{t_{i+1}} \mid \boldsymbol{Z}_{t_i} = \boldsymbol{z}),$$
$$\boldsymbol{\Omega}_{S,i} = \boldsymbol{\Omega}_S(t_i, \boldsymbol{z}; t_{i+1}) = \mathrm{E}(\boldsymbol{R}_{t_{i+1}}\boldsymbol{R}'_{t_{i+1}} \mid \boldsymbol{Z}_{t_i} = \boldsymbol{z}).$$

The set of admissible portfolios Φ is again given by portfolios that satisfy the following two conditions. First, admissible portfolios are self-financing, so that they satisfy $V_{\boldsymbol{\phi},t_i} = \boldsymbol{\imath}'\boldsymbol{\phi}_{t_i}$, for $i = 0, \ldots, n$, and $V_{\boldsymbol{\phi},t_{i+1}} = \boldsymbol{\phi}'_{t_i}\boldsymbol{R}_{t_{i+1}}$, for $i = 0, \ldots, n-1$. Secondly, they satisfy the property $E\left(V^2_{\boldsymbol{\phi},t_{i+1}}\right) < \infty$, for $i = 0, \ldots, n-1$.

A.1.2 The one-period frontier

In discrete time the matrices of second conditional moments $\boldsymbol{\Omega}_S(t_i, \boldsymbol{z}; t_{i+1})$ are assumed to be nonsingular for all t_i, $i = 0, \ldots, n-1$ and $\boldsymbol{z}$. So, again the case of a riskless asset is covered, just as the case when all assets are risky. The one-period solution is given by

$$s^2_{\mathrm{LSR},i} = s^2_{\mathrm{LSR}}(t_i, \boldsymbol{z}; t_{i+1}) = \frac{1}{\boldsymbol{\imath}'\boldsymbol{\Omega}^{-1}_{S,i}\boldsymbol{\imath}}, \tag{A.1}$$

$$m_{\mathrm{LSR},i} = m_{\mathrm{LSR}}(t_i, \boldsymbol{z}; t_{i+1}) = \frac{\boldsymbol{m}'_{S,i}\boldsymbol{\Omega}^{-1}_{S,i}\boldsymbol{\imath}}{\boldsymbol{\imath}'\boldsymbol{\Omega}^{-1}_{S,i}\boldsymbol{\imath}}, \tag{A.2}$$

$$F^2_i = F^2(t_i, \boldsymbol{z}; t_{i+1}) = \boldsymbol{m}'_{S,i}\left(\boldsymbol{\Omega}^{-1}_{S,i} - \frac{\boldsymbol{\Omega}^{-1}_{S,i}\boldsymbol{\imath}\boldsymbol{\imath}'\boldsymbol{\Omega}^{-1}_{S,i}}{\boldsymbol{\imath}'\boldsymbol{\Omega}^{-1}_{S,i}\boldsymbol{\imath}}\right)\boldsymbol{m}_{S,i}. \tag{A.3}$$

[9] When there can be no confusion, we use the short notation, such as, $\boldsymbol{m}_{S,i}$. In other cases, where we need be more explicit, we use the other notation, such as $\boldsymbol{m}_S(t_i, \boldsymbol{z}; t_{i+1})$.

Furthermore, the one-period MV efficient MR and LSR portfolios are given by

$$\boldsymbol{\phi}_{\text{MR}}(t_i, \boldsymbol{z}; t_{i+1}) = \frac{\boldsymbol{\Omega}_{S,i}^{-1}\boldsymbol{m}_{S,i}}{\boldsymbol{\imath}'\boldsymbol{\Omega}_{S,i}^{-1}\boldsymbol{m}_{S,i}}, \qquad \boldsymbol{\phi}_{\text{LSR}}(t_i, \boldsymbol{z}; t_{i+1}) = \frac{\boldsymbol{\Omega}_{S,i}^{-1}\boldsymbol{\imath}}{\boldsymbol{\imath}'\boldsymbol{\Omega}_{S,i}^{-1}\boldsymbol{\imath}}. \tag{A.4}$$

When $\boldsymbol{\Sigma}_{S,i} = \boldsymbol{\Omega}_{S,i} - \boldsymbol{m}_{S,i}\boldsymbol{m}_{S,i}'$ is nonsingular we find

$$\boldsymbol{\phi}_{\text{MR}}(t_i, \boldsymbol{z}; t_{i+1}) = \frac{\boldsymbol{\Sigma}_{S,i}^{-1}\boldsymbol{m}_{S,i}}{\boldsymbol{\imath}'\boldsymbol{\Sigma}_{S,i}^{-1}\boldsymbol{m}_{S,i}}, \qquad \boldsymbol{\phi}_{\text{GMV}}(t_i, \boldsymbol{z}; t_{i+1}) = \frac{\boldsymbol{\Sigma}_{S,i}^{-1}\boldsymbol{\imath}}{\boldsymbol{\imath}'\boldsymbol{\Sigma}_{S,i}^{-1}\boldsymbol{\imath}}.$$

A.1.3 MV efficiency in discrete time

In discrete time, the value function (4) applies for $t = t_i$, $i = 0, \ldots, n$ and Bellman's principle of optimality now requires both

$$J(t_i, \boldsymbol{z}, x) = \min_{\boldsymbol{\phi}_{t_i}} \mathrm{E}(J(t_{i+1}, \boldsymbol{Z}_{i+1}, \boldsymbol{\phi}_{t_i}'\boldsymbol{R}_{t_{i+1}}) \mid \boldsymbol{Z}_{t_i} = \boldsymbol{z}, \boldsymbol{\imath}'\boldsymbol{\phi}_{t_i} = x), \tag{A.5}$$

for $i = 0, \ldots, n-1$, and the boundary condition (14) to hold true.

Theorem 2. *For $\mathcal{T} = \{t_0, \ldots, t_n\}$, with $0 = t_0 < \ldots < t_n = T$ the value function (3) is of the form (4) and the dynamic MV parameters s_{LSR}^2, m_{LSR} and F^2: $\mathcal{T} \times \mathbb{R}^M \mapsto \mathbb{R}$ are recursively defined by the following system of equations*

$$s_{LSR}^2 = s_{LSR}^2(t_i, \boldsymbol{z}; t_n) = \frac{1}{\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}}, \tag{A.6}$$

$$m_{LSR} = m_{LSR}(t_i, \boldsymbol{z}; t_n) = \frac{\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}\boldsymbol{p}_i}{\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}}, \tag{A.7}$$

$$F^2 = F^2(t_i, \boldsymbol{z}; t_n) = u_i^2 + \boldsymbol{p}_i'\left(\boldsymbol{Q}_i^{-1} - \frac{\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}}{\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}}\right)\boldsymbol{p}_i, \tag{A.8}$$

and

$$\begin{aligned}
\boldsymbol{Q}_i &= \mathrm{E}(s_{LSR}^2(t_{i+1}, \boldsymbol{Z}_{t_{i+1}}; t_n)\boldsymbol{R}_{t_{i+1}}\boldsymbol{R}_{t_{i+1}}' \mid \boldsymbol{Z}_{t_i} = \boldsymbol{z}), \\
\boldsymbol{p}_i &= \mathrm{E}(m_{LSR}(t_{i+1}, \boldsymbol{Z}_{t_{i+1}}; t_n)\boldsymbol{R}_{t_{i+1}} \mid \boldsymbol{Z}_{t_i} = \boldsymbol{z}), \\
u_i^2 &= \mathrm{E}(F^2(t_{i+1}, \boldsymbol{Z}_{t_{i+1}}; t_n) \mid \boldsymbol{Z}_{t_i} = \boldsymbol{z}),
\end{aligned}$$

with boundary conditions $m_{LSR}(t_n, \boldsymbol{z}; t_n) = s_{LSR}^2(t_n, \boldsymbol{z}; t_n) = 1 - F^2(t_n, \boldsymbol{z}; t_n) = 1$.

The portfolio weights $\boldsymbol{\phi}_{t_i}$: $\mathcal{T} \times \mathbb{R}^M \mapsto \mathbb{R}^N$ for the strategy that minimizes (12) are given *by*

$$\boldsymbol{\phi}_{t_i} = \frac{\boldsymbol{Q}_i^{-1}\boldsymbol{p}_i}{\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}\boldsymbol{p}_i} C \frac{m_{LSR}}{s_{LSR}^2} + \frac{\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}}{\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}} \left(V_{\boldsymbol{\phi},t_i} - C\frac{m_{LSR}}{s_{LSR}^2} \right). \tag{A.9}$$

The proof is given in Appendix A.3.

A.1.3.1 The deterministic case

In the deterministic case, when there are no state variables, we find $\boldsymbol{Q}_i = \boldsymbol{\Omega}_{S,i} s_{\mathrm{LSR}}^2(t_{i+1}; t_n)$, $\boldsymbol{p}_i = \boldsymbol{m}_{S,i} m_{\mathrm{LSR}}(t_{i+1}; t_n)$, and $u_i^2 = F^2(t_{i+1}; t_n)$. So, using (A.1), (A.2) and (A.3), the conditions (A.6), (A.7), and (A.8) reduce to

$$\begin{aligned} s_{\mathrm{LSR}}^2 &= s_{\mathrm{LSR},i}^2 s_{\mathrm{LSR}}^2(t_{i+1}; t_n), \\ m_{\mathrm{LSR}} &= m_{\mathrm{LSR},i} m_{\mathrm{LSR}}(t_{i+1}; t_n), \\ F^2 &= F^2(t_{i+1}; t_n) + F_i^2 \frac{m_{\mathrm{LSR}}^2(t_{i+1}; t_n)}{s_{\mathrm{LSR}}^2(t_{i+1}; t_n)}, \end{aligned}$$

respectively. In that case we find

$$s_{\mathrm{LSR}}^2 = \prod_{j=i}^{n-1} s_{\mathrm{LSR},j}^2, \tag{A.10}$$

$$m_{\mathrm{LSR}} = \prod_{j=i}^{n-1} m_{\mathrm{LSR},j}, \tag{A.11}$$

$$F^2 = \sum_{j=i}^{n-1} \left\{ F_j^2 \prod_{k=j+1}^{n-1} \frac{m_{\mathrm{LSR}}^2(t_k; t_n)}{s_{\mathrm{LSR}}^2(t_k; t_n)} \right\}, \tag{A.12}$$

and the portfolio weights are given by

$$\begin{aligned} \phi_{t_i} &= \frac{\boldsymbol{\Omega}_{S,i}^{-1}\boldsymbol{m}_{S,i}}{\boldsymbol{\imath}'\boldsymbol{\Omega}_{S,i}^{-1}\boldsymbol{m}_{S,i}} C \prod_{j=i}^{n-1}\left(\frac{m_{\mathrm{LSR},j}}{s_{\mathrm{LSR},j}^2}\right) + \frac{\boldsymbol{\Omega}_{S,i}^{-1}\boldsymbol{\imath}}{\boldsymbol{\imath}'\boldsymbol{\Omega}_{S,i}^{-1}\boldsymbol{\imath}} \left\{ V_{\phi,t_i} - C\prod_{j=i}^{n-1}\left(\frac{m_{\mathrm{LSR},j}}{s_{\mathrm{LSR},j}^2}\right) \right\} \qquad \text{(A.13)} \\ &= \boldsymbol{\phi}_{\mathrm{MR}}(t_i; t_{i+1}) C \frac{m_{\mathrm{LSR}}}{s_{\mathrm{LSR}}^2} + \boldsymbol{\phi}_{\mathrm{LSR}}(t_i; t_{i+1}) \left(V_{\boldsymbol{\phi},t_i} - C\frac{m_{\mathrm{LSR}}}{s_{\mathrm{LSR}}^2} \right), \end{aligned}$$

where we used the one-period portfolios (A.4).

Again, the LSR and MR strategies are uniformly LSR and MR, respectively, and hence they are uniformly MV efficient, since they do not depend on T:

$$\phi_{\text{LSR},t_i} = \phi_{\text{LSR}}(t_i;t_{i+1})V_{\text{LSR},t_i} \tag{A.14}$$

$$\phi_{\text{MR},t_i} = \phi_{\text{MR}}(t_i;t_{i+1})\prod_{j=0}^{i-1}\left(\frac{s^2_{\text{LSR},j}}{m_{\text{LSR},j}}\right) + \phi_{\text{LSR}}(t_i;t_{i+1})\left\{V_{\text{MR},t_i} - \prod_{j=0}^{i-1}\left(\frac{s^2_{\text{LSR},j}}{m_{\text{LSR},j}}\right)\right\}. \tag{A.15}$$

Thus we find that uniform efficiency is not restricted to continuous trading. Also in discrete time, with a deterministic market, MV efficient strategies span MV frontiers of all terms T, whether or not there is a risk-free asset. Both short-term and long-term investors agree about the MV efficient strategies.

A.2 Proof of Theorem 1

Using the Hamilton-Jacobi-Bellman equation (13) and the optimal value function (4) we find

$$A(t,\boldsymbol{z},x) = \begin{pmatrix} x \\ -C \end{pmatrix}' (\boldsymbol{A}_t + \boldsymbol{A}_{\boldsymbol{z}} + \boldsymbol{A}_{\boldsymbol{zz}}) \begin{pmatrix} x \\ -C \end{pmatrix},$$

where

$$\boldsymbol{A}_t = \begin{pmatrix} \frac{\partial s^2_{\text{LSR}}}{\partial t} & \frac{\partial m_{\text{LSR}}}{\partial t} \\ \frac{\partial m_{\text{LSR}}}{\partial t} & -\frac{\partial F^2}{\partial t} \end{pmatrix}, \qquad \boldsymbol{A}_{\boldsymbol{z}} = \begin{pmatrix} \boldsymbol{\mu}_{\boldsymbol{z}}' \frac{\partial s^2_{\text{LSR}}}{\partial \boldsymbol{z}} & \boldsymbol{\mu}_{\boldsymbol{z}}' \frac{\partial m_{\text{LSR}}}{\partial \boldsymbol{z}} \\ \boldsymbol{\mu}_{\boldsymbol{z}}' \frac{\partial m_{\text{LSR}}}{\partial \boldsymbol{z}} & -\boldsymbol{\mu}_{\boldsymbol{z}}' \frac{\partial F^2}{\partial \boldsymbol{z}} \end{pmatrix},$$

and

$$\boldsymbol{A}_{\boldsymbol{zz}} = 1/2 \begin{pmatrix} \operatorname{tr}\left\{\boldsymbol{\Sigma}_{\boldsymbol{z}} \frac{\partial^2 s^2_{\text{LSR}}}{\partial \boldsymbol{z} \partial \boldsymbol{z}'}\right\} & \operatorname{tr}\left\{\boldsymbol{\Sigma}_{\boldsymbol{z}} \frac{\partial^2 m_{\text{LSR}}}{\partial \boldsymbol{z} \partial \boldsymbol{z}'}\right\} \\ \operatorname{tr}\left\{\boldsymbol{\Sigma}_{\boldsymbol{z}} \frac{\partial^2 m_{\text{LSR}}}{\partial \boldsymbol{z} \partial \boldsymbol{z}'}\right\} & -\operatorname{tr}\left\{\boldsymbol{\Sigma}_{\boldsymbol{z}} \frac{\partial^2 F^2}{\partial \boldsymbol{z} \partial \boldsymbol{z}'}\right\} \end{pmatrix}.$$

To compute $B(t,x,\boldsymbol{z})$, the Lagrangian is given by

$$L(\phi_t,\lambda) = \phi_t'(\boldsymbol{\mu}_S J_x + \boldsymbol{\Sigma}_{SZ}\boldsymbol{J}_{\boldsymbol{z}x}) + \frac{1}{2}\phi_t'\boldsymbol{\Sigma}_S\phi J_{xx} - \lambda(\boldsymbol{\imath}'\phi_t - x),$$

with first order conditions

$$\begin{aligned} \boldsymbol{\mu}_S J_x + \boldsymbol{\Sigma}_{SZ}\boldsymbol{J}_{\boldsymbol{z}x} + \boldsymbol{\Sigma}_S\phi_t J_{xx} - \lambda\boldsymbol{\imath} &= 0, \\ \boldsymbol{\imath}'\phi_t &= x. \end{aligned}$$

The first order conditions can be reexpressed as

$$\begin{pmatrix} \boldsymbol{\Sigma}_S & \boldsymbol{\imath} \\ \boldsymbol{\imath} & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{\phi}_t \\ -\frac{\lambda}{J_{xx}} \end{pmatrix} = \begin{pmatrix} -\boldsymbol{\mu}_S \frac{J_x}{J_{xx}} - \boldsymbol{\Sigma}_{SZ} \frac{\boldsymbol{J_{zx}}}{J_{xx}} \\ x \end{pmatrix}.$$

Notice that

$$J_x = 2(s^2_{lsr}, m_{lsr}) \begin{pmatrix} x \\ -C \end{pmatrix}, \qquad \boldsymbol{J_{zx}} = 2 \left(\frac{\partial s^2_{lsr}}{\partial \boldsymbol{z}}, \frac{\partial m_{lsr}}{\partial \boldsymbol{z}} \right) \begin{pmatrix} x \\ -C \end{pmatrix}, \qquad J_{xx} = 2 s^2_{\mathrm{LSR}}.$$

Consequently, using (6), $\boldsymbol{\phi}_t$ is given by

$$\boldsymbol{\phi}_t = \{ \boldsymbol{h}(1,0) - \boldsymbol{H\Psi} \} \begin{pmatrix} x \\ -C \end{pmatrix},$$

where

$$\boldsymbol{\Psi} = \boldsymbol{\Psi}(t, \boldsymbol{z}; T) = s^{-2}_{\mathrm{LSR}} \left\{ \boldsymbol{\mu}_S (s^2_{\mathrm{LSR}}, m_{\mathrm{LSR}}) + \boldsymbol{\Sigma}_{SZ} \left(\frac{\partial s^2_{\mathrm{LSR}}}{\partial \boldsymbol{z}}, \frac{\partial m_{\mathrm{LSR}}}{\partial \boldsymbol{z}} \right) \right\},$$

which amounts to (18). Furthermore,

$$B(t, \boldsymbol{z}, x) = \begin{pmatrix} x \\ -C \end{pmatrix}' \boldsymbol{B_\phi} \begin{pmatrix} x \\ -C \end{pmatrix},$$

$$\boldsymbol{B_\phi} = s^2_{\mathrm{LSR}} \left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \boldsymbol{h}' \boldsymbol{\Psi} + \boldsymbol{\Psi}' \boldsymbol{h}(1,0) + \boldsymbol{h}' \boldsymbol{\Sigma}_S \boldsymbol{h} \begin{pmatrix} 1 \\ 0 \end{pmatrix} (1,0) - \boldsymbol{\Psi}' \boldsymbol{H\Psi} \right\}.$$

As (13) holds for all x, we find

$$\boldsymbol{A}_t + \boldsymbol{A_z} + \boldsymbol{A_{zz}} + \boldsymbol{B_\phi} = 0, \tag{A.16}$$

which amounts to the three conditions (15), (16), and (17). The boundary conditions follow from (14). The Verification Theorem of stochastic optimal control theory now implies that the value function is of the form (4) and the optimal strategy is given by (18). □

A.3 Proof of Theorem 2

Using the Bellman equation (A.5) and the optimal value function (4) we find

$$J(t_i, \boldsymbol{z}, x) = \min_{\boldsymbol{\phi}_{t_i}} \left\{ \begin{pmatrix} \boldsymbol{\phi}_{t_i} \\ -C \end{pmatrix}' \begin{pmatrix} \boldsymbol{Q}_i & \boldsymbol{p}_i \\ \boldsymbol{p}_i' & 1 - u_i^2 \end{pmatrix} \begin{pmatrix} \boldsymbol{\phi}_{t_i} \\ -C \end{pmatrix} \mid \boldsymbol{\imath}' \boldsymbol{\phi}_{t_i} = x \right\}.$$

The solution is found for

$$\boldsymbol{\phi}_{t_i} = \left\{ \frac{\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}}{\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}}, -\left(\boldsymbol{Q}_i^{-1} - \frac{\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}}{\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}}\right)\boldsymbol{p}_i \right\} \begin{pmatrix} x \\ -C \end{pmatrix},$$

which amounts to (A.9). For the value function we thus find

$$\begin{aligned} J(t_i, \boldsymbol{z}, x) &= \begin{pmatrix} x \\ -C \end{pmatrix}' \begin{pmatrix} \frac{\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}}{\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}} & -\left(\boldsymbol{Q}_i^{-1} - \frac{\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}}{\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}}\right)\boldsymbol{p}_i \\ 0 & 1 \end{pmatrix}' \begin{pmatrix} \boldsymbol{Q}_i & \boldsymbol{p}_i \\ \boldsymbol{p}_i' & 1 - u_i^2 \end{pmatrix} \times \\ &\qquad \begin{pmatrix} \frac{\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}}{\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}} & -\left(\boldsymbol{Q}_i^{-1} - \frac{\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}}{\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}}\right)\boldsymbol{p}_i \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ -C \end{pmatrix} \\ &= \begin{pmatrix} x \\ -C \end{pmatrix}' \begin{pmatrix} \frac{1}{\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}} & \frac{\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}\boldsymbol{p}_i}{\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}} \\ \frac{\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}\boldsymbol{p}_i}{\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}} & 1 - u_i^2 - \boldsymbol{p}_i'\left(\boldsymbol{Q}_i^{-1} - \frac{\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}}{\boldsymbol{\imath}'\boldsymbol{Q}_i^{-1}\boldsymbol{\imath}}\right)\boldsymbol{p}_i \end{pmatrix} \begin{pmatrix} x \\ -C \end{pmatrix}. \end{aligned}$$

Therefore, the value function is given by (4), where the mean variance parameters satisfy the system of recursive equations (A.6), (A.7), and (A.8). Furthermore, the boundary conditions follow from (14). It follows that the optimal strategy is given by (A.9). □

A.4 Derivation of s_{LSR}^2 and m_{LSR} in Example 1

Conditions (15) and (16) amount to

$$\frac{\partial s_{\text{LSR}}^2}{\partial t} + \kappa(\beta_o - z)\frac{\partial s_{\text{LSR}}^2}{\partial z} + \frac{1}{2}\sigma_o^2\frac{\partial^2 s_{\text{LSR}}^2}{\partial z^2} + s_{\text{LSR}}^2(2z + \alpha^2 - \gamma^2) = 0, \tag{A.17}$$

$$\frac{\partial m_{\text{LSR}}}{\partial t} + \kappa(\beta_o - z)\frac{\partial m_{\text{LSR}}}{\partial z} + \frac{1}{2}\sigma_o^2\frac{\partial^2 m_{\text{LSR}}}{\partial z^2} + m_{\text{LSR}}(z - \gamma^2) = 0. \tag{A.18}$$

Assume the solution is of the form $s_{\text{LSR}}^2 = \exp\{p_s(t) + q_s(t)z\}$ and $m_{\text{LSR}} = \exp\{p_m(t) + q_m(t)z\}$, for C^2 functions $p_s = p_s(t)$, $q_s = q_s(t)$, $p_m = p_m(t)$ and $q_m = q_m(t) : \mathbb{R}_+ \mapsto \mathbb{R}$. The boundary conditions imply $p_s(T) = q_s(T) = p_m(T) = q_m(T) = 0$. In that case we find

$$\frac{\partial s_{\text{LSR}}^2}{\partial t} = \{p_s' + q_s' z\}s_{\text{LSR}}^2, \qquad \frac{\partial s_{\text{LSR}}^2}{\partial z} = q_s s_{\text{LSR}}^2, \qquad \frac{\partial^2 s_{\text{LSR}}^2}{\partial z^2} = q_s^2 s_{\text{LSR}}^2,$$

$$\frac{\partial m_{\text{LSR}}}{\partial t} = \{p_m' + q_m' z\}m_{\text{LSR}}, \qquad \frac{\partial m_{\text{LSR}}}{\partial z} = q_m m_{\text{LSR}}, \qquad \frac{\partial^2 m_{\text{LSR}}}{\partial z^2} = q_m^2 m_{\text{LSR}}.$$

Consequently, (A.17) and (A.18) amount to

$$s_{\text{LSR}}^2 \left\{ p_s' + q_s' z + \kappa(\beta_o - z) q_s + \frac{1}{2}\sigma_o^2 q_s^2 + 2z + \alpha^2 - \gamma^2 \right\} = 0, \tag{A.19}$$

$$m_{\text{LSR}} \left\{ p_m' + q_m' z + \kappa(\beta_o - z) q_m + \frac{1}{2}\sigma_o^2 q_m^2 + z - \gamma^2 \right\} = 0. \tag{A.20}$$

As both s_{LSR}^2 and m_{LSR} are positive, and (A.19) and (A.20) hold for all z, we find

$$q_s' + 2 - \kappa q_s = 0, \tag{A.21}$$

$$p_s' + \kappa\beta_o q_s + \frac{1}{2}\sigma_o^2 q_s^2 + \alpha^2 - \gamma^2 = 0, \tag{A.22}$$

$$q_m' + 1 - \kappa q_s = 0, \tag{A.23}$$

$$p_m' + \kappa\beta_o q_m + \frac{1}{2}\sigma_o^2 q_m^2 - \gamma^2 = 0. \tag{A.24}$$

Solving (A.21) and (A.23) and applying the boundary conditions, we obtain

$$q_s = \frac{2}{\kappa}\left\{1 - e^{-\kappa(T-t)}\right\}, \qquad q_m = \frac{1}{\kappa}\left\{1 - e^{-\kappa(T-t)}\right\}.$$

Substituting this solution in (A.22) and (A.24) and integrating we find

$$p_s = \left(2\beta_o + \alpha^2 - \gamma^2 + \frac{2\sigma_o^2}{\kappa^2}\right)(T-t) - \left(\frac{2\beta_0}{\kappa} + \frac{4\sigma_o^2}{\kappa^3}\right)\left\{1 - e^{-\kappa(T-t)}\right\} + \frac{\sigma_o^2}{\kappa^3}\left\{1 - e^{-2\kappa(T-t)}\right\},$$

$$p_m = \left(\beta_o - \gamma^2 + \frac{\sigma_o^2}{2\kappa^2}\right)(T-t) - \left(\frac{\beta_0}{\kappa} + \frac{\sigma_o^2}{\kappa^3}\right)\left\{1 - e^{-\kappa(T-t)}\right\} + \frac{\sigma_o^2}{4\kappa^3}\left\{1 - e^{-2\kappa(T-t)}\right\}.$$

So, we can conclude that s^2_{LSR} and m_{LSR} are indeed of the form specified above, as given in (27) and (28):

$$\begin{aligned} s^2_{\text{LSR}} &= \exp\left[\left(2\beta_o + \alpha^2 - \gamma^2 + \frac{2\sigma_o^2}{\kappa^2}\right)(T-t) - \left(\frac{2(\beta_0 - z)}{\kappa} + \frac{4\sigma_o^2}{\kappa^3}\right)\left\{1 - e^{-\kappa(T-t)}\right\}\right. \\ &\qquad \left. + \frac{\sigma_o^2}{\kappa^3}\left\{1 - e^{-2\kappa(T-t)}\right\}\right], \\ m_{\text{LSR}} &= \exp\left[\left(\beta_o - \gamma^2 + \frac{\sigma_o^2}{2\kappa^2}\right)(T-t) - \left(\frac{\beta_0 - z}{\kappa} + \frac{\sigma_o^2}{\kappa^3}\right)\left\{1 - e^{-\kappa(T-t)}\right\}\right. \\ &\qquad \left. + \frac{\sigma_o^2}{4\kappa^3}\left\{1 - e^{-2\kappa(T-t)}\right\}\right]. \end{aligned}$$

We also find that $m^2_{\text{LSR}}/s^2_{\text{LSR}}$ is deterministic, since it does not depend on z. Consequently, condition (17) implies that also F^2 is deterministic, as in (29):

$$F^2 = \gamma^2 \int_t^T \frac{m^2_{\text{LSR}}(u, z; T)}{s^2_{\text{LSR}}(u, z; T)}\,\mathrm{d}u.$$

The MR value $C = s^2_{\text{LSR}}(0, Z_0; T)/m_{\text{LSR}}(0, Z_0; T)$ produces the result in Example 1.

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