Summer Course Mathematics

Facts Thursday

1. Rules for shifting the graph of a function $f(x)$:
   - If $y = f(x)$ is replaced by $y = f(x) + c$ with $c > 0$, the graph is moved upwards by c units.
   - If $y = f(x)$ is replaced by $y = f(x) - c$ with $c > 0$, the graph is moved downwards by c units.
   - If $y = f(x)$ is replaced by $y = f(x + c)$ with $c > 0$, the graph is moved c units to the left.
   - If $y = f(x)$ is replaced by $y = f(x - c)$ with $c > 0$, the graph is moved c units to the right.
   - If $y = f(x)$ is replaced by $y = cf(x)$ with $c > 0$, the graph is stretched vertically with factor c.
   - If $y = f(x)$ is replaced by $y = -f(x)$, the graph is reflected through the x-axis.
   - If $y = f(x)$ is replaced by $y = f(-x)$, the graph is reflected through the y-axis.

2. Functions can be added, subtracted, multiplied and divided:
   - $(f + g)(x) = f(x) + g(x)$.
   - $(f - g)(x) = f(x) - g(x)$.
   - $(fg)(x) = f(x) \cdot g(x)$.
   - $(f/g)(x) = \dfrac{f(x)}{g(x)}$.

3. The slope of a function $f(x)$ for a certain x is defined as the slope of the tangent to graph of $f(x)$ at the point $(x, f(x))$.

   The slope of the tangent to the graph of $f(x)$ at a certain x is called the derivative of $f(x)$ at x and is denoted by $f'(x)$:

   $$f'(x) = \text{slope of the tangent to the graph of } f(x) \text{ at } (x, f(x)).$$

   The derivative $f'(x)$ of $f(x)$ is defined as a limit:

   $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}.$$

4. Various notation for the derivative of a function $y = f(x)$ used in mathematics:
   - $f'(x)$.
   - $\dfrac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{d}y/\mathrm{d}x$.
   - $\dfrac{\mathrm{d}f(x)}{\mathrm{d}x} = \mathrm{d}f(x)/\mathrm{d}x$.
   - $\dfrac{\mathrm{d}}{\mathrm{d}x} f(x)$.

5. Consider a function f that is defined in an interval I and two numbers x_1 and x_2 in that interval.

   - If $f(x_2) \geq f(x_1)$ whenever $x_2 > x_1$, then f is increasing in I.
   - If $f(x_2) > f(x_1)$ whenever $x_2 > x_1$, then f is strictly increasing in I.
   - If $f(x_2) \leq f(x_1)$ whenever $x_2 > x_1$, then f is decreasing in I.
   - If $f(x_2) < f(x_1)$ whenever $x_2 > x_1$, then f is strictly decreasing in I.

6. The derivative $f'(x)$ of a function $f(x)$ gives the slope the function $f(x)$ for any value of x.

   - A positive slope is equivalent to an increasing function.
   - A negative slope is equivalent to a decreasing function.

   Consequence:

$$\begin{array}{lcl} f'(x) \geq 0 \text{ for all } x \text{ in the interval } I & \iff & f \text{ is increasing in } I \\ f'(x) \leq 0 \text{ for all } x \text{ in the interval } I & \iff & f \text{ is decreasing in } I \\ f'(x) = 0 \text{ for all } x \text{ in the interval } I & \iff & f \text{ is constant in } I \end{array}$$

7. Rules for differentiation:

$$\begin{array}{llll} f(x) = A & \Rightarrow & f'(x) = 0 & \text{(Constant)} \\ f(x) = A + g(x) & \Rightarrow & f'(x) = g'(x) & \text{(Additive constant)} \\ f(x) = Ag(x) & \Rightarrow & f'(x) = Ag'(x) & \text{(Multiplicative constant)} \\ f(x) = x^a & \Rightarrow & f'(x) = ax^{a-1} & \text{(Power rule)} \\ f(x) = p(x) + q(x) & \Rightarrow & f'(x) = p'(x) + q'(x) & \text{(Sum rule)} \\ f(x) = p(x) \cdot q(x) & \Rightarrow & f'(x) = p'(x) \cdot q(x) + p(x) \cdot q'(x) & \text{(Product rule)} \\ f(x) = \dfrac{p(x)}{q(x)} & \Rightarrow & f'(x) = \dfrac{p'(x) \cdot q(x) - p(x) \cdot q'(x)}{(q(x))^2} & \text{(Quotient rule)} \\ f(x) = g(u(x)) & \Rightarrow & f'(x) = g'(u(x)) \cdot u'(x) & \text{(Chain rule)} \\ f(x) = \mathrm{e}^{g(x)} & \Rightarrow & f'(x) = \mathrm{e}^{g(x)} g'(x) & \text{(Exponential function)} \end{array}$$