A comparison between LQR control for a long string of SISO systems and LQR control of the infinite spatially invariant version

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Abstract

In this paper we consider a long string of SISO systems which in the limit becomes a scalar infinite spatially invariant system. We compare the LQR control for long-but-finite strings with the the LQR control for the corresponding infinite strings. We give analytic and numerical examples where these are quite different and we investigate the cause. In addition, we obtain sufficient conditions for the LQR solutions to be similar as the length of the string increases.

Key words: Linear systems, infinite-dimensional systems, LQR control, spatially invariant systems, spatially distributed systems.

1 Introduction

In Bamieh et al [1] a general class of spatially invariant systems was introduced as a useful model for many applications. While large scale finite-dimensional systems are cumbersome to treat, spatially invariant systems are easier to analyze mathematically. In [12] Jovanović and Bamieh pointed out the shortcomings of previous models of platoons of vehicles in Levine and Athans [16], Melzer and Kuo [17], [18], which were due to lack of exponential stabilizability or detectability of the infinite platoon model. They also studied the LQR problem for an example of infinite spatially invariant string of vehicles and argued that the LQR control of the infinite model reflected well the behavior of the long-but-finite vehicular platoons described in [16–18]. In view of the attractive mathematical features of the class of spatially invariant systems and their applications (see [1]), it is important to investigate when spatially invariant systems serve as good models for long-but-finite strings. This paper serves as a first step in addressing this very complex question.

We consider a long-but-finite string of $2N+1$ scalar autonomous units

$$\dot{z}_r(t) = \sum_{l=-N}^{N} a_l z_{r-l}(t) + \sum_{l=-N}^{N} b_l u_{r-l}(t), \qquad (1)$$

$$y_r(t) = \sum_{l=-N}^{N} c_l z_{r-l}(t), \quad -N \le r \le N,\ t \ge 0,$$

where only finitely many $s \ll N$ of the coefficients $a_l, b_l, c_l \in \mathbb{C}$ are nonzero and z_k, y_k and u_k are set to zero for $|k| > N$. We remark that the structure of the first s and the last s units of the long-but-finite string (1) are different from the structure of the other units. We compare the LQR control for (1) with LQR control of the corresponding infinite string, which is a scalar spatially invariant system. By means of two analytical counterexamples and numerical counterexamples we show that that finite and infinite strings can exhibit quite different behavior. So for this class of scalar systems the infinite case is not always a useful paradigm to understand the long-but-finite case. More importantly, we investigate the mathematical underpinnings of the LQR control problem for both the finite and infinite case. We give sufficient conditions under which different types of long-but-finite strings (such as circular configurations and systems for which two out of the three defining operators are constants) exhibit similar behavior as the corresponding infinite strings.

Clearly, it is important to clarify which properties one is considering and whether or not the infinite model does serve

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as a useful indicator for these properties. Among the many properties one could consider of the LQR solution we focus on the following two properties of the closed-loop generator A_{cl}: the *growth bound* which equals the spectral bound $\omega_{cl} = \sup\{\mathrm{Re}(\lambda) : \lambda \in \sigma(A_{cl})\}$ (since A_{cl} is bounded) and a *transient bound* M_ω, which is such that for any $\omega > \omega_{cl}$ we have $\|e^{A_{cl}t}\| \le M_\omega e^{\omega t}$ (see [4, Theorem 2.1.6]). The model (1) with state space $\mathbb{C}^{2N+1}$ can also be written in a compact form $\Sigma(\mathbf{A}_N, \mathbf{B}_N, \mathbf{C}_N, 0)$

$$\begin{aligned} \dot{\mathbf{z}}_N(t) &= \mathbf{A}_N \mathbf{z}_N(t) + \mathbf{B}_N \mathbf{u}_N(t), \\ \mathbf{y}_N(t) &= \mathbf{C}_N \mathbf{z}_N(t), \ t \ge 0, \end{aligned} \tag{2}$$

where $\mathbf{u}, \mathbf{y}, \mathbf{z}$ are column vectors of size $2N+1$, e.g., $\mathbf{z}_N(t) = \begin{bmatrix} z_{-N}(t) & z_{-N+1}(t) & \cdots & z_N(t) \end{bmatrix}^T$ and $\mathbf{A}_N, \mathbf{B}_N, \mathbf{C}_N$ are $(2N+1) \times (2N+1)$ Toeplitz matrices. It is well known that large Toeplitz matrices have bad numerical properties and so simulations are not in general a reliable way to investigate the properties of Toeplitz systems (see Böttcher and Silverman [3]). Consequently, it is important to analyze these systems analytically. Now the limit as $N \to \infty$ produces a system that is amenable to mathematical computations. This infinite-dimensional string falls into the class of *spatially invariant systems* introduced in [1] and is given by

$$\dot{z}_r(t) = \sum_{l \in \mathbb{Z}} a_l z_{r-l}(t) + \sum_{l \in \mathbb{Z}} b_l u_{r-l}(t), \tag{3}$$

$$y_r(t) = \sum_{l \in \mathbb{Z}} c_l z_{r-l}(t), \quad r \in \mathbb{Z}, \quad t \ge 0, \tag{4}$$

where $a_l, b_l, c_l \in \mathbb{C}$ and $z_r(t)$, $u_r(t)$ and $y_r(t) \in \mathbb{C}$ are the state, the input and the output vectors, respectively, at time $t \ge 0$ and spatial point $r \in \mathbb{Z}$. As in [5,6] we can formulate (3), (4) as a standard state linear system $\Sigma(A, B, C, 0)$

$$\begin{aligned} \dot{z}(t) &= (Az)(t) + (Bu)(t), \\ y(t) &= (Cz)(t), \ t \ge 0, \end{aligned} \tag{5}$$

with the state, the input and the output spaces (Z, $U$and Y, respectively) are equal to $\ell_2(\mathbb{Z}, \mathbb{C})$. A, B, C are Laurent operators (see Appendix).

Taking Fourier transforms of the system equations (5), we obtain the state linear system $\Sigma(\breve{A}, \breve{B}, \breve{C}, 0)$

$$\begin{aligned} \dot{\breve{z}}(t) = \mathfrak{F}\dot{z}(t) &= \breve{A}\breve{z}(t) + \breve{B}\breve{u}(t), \\ \breve{y}(t) = \mathfrak{F}y(t) &= \breve{C}\breve{z}(t), \ t \ge 0. \end{aligned} \tag{6}$$

Note that our standing assumption is that only finitely many of the coefficients are nonzero which means that $\breve{A}(e^{j\theta}), \breve{B}(e^{j\theta}), \breve{C}(e^{j\theta})$ are uniformly continuous in θ on $[0, 2\pi]$ and $\breve{A}, \breve{B}, \breve{C} \in \mathbf{L}_\infty(\partial\mathbb{D}; \mathbb{C})$. Hence $\breve{A}, \breve{B}, \breve{C}$ define bounded operators on $\mathbf{L}_2(\partial\mathbb{D}; \mathbb{C})$. The system $\Sigma(A, B, C, 0)$ is isometrically isomorphic to $\Sigma(\breve{A}, \breve{B}, \breve{C}, 0)$ with the state space, input and output spaces $\mathbf{L}_2(\partial\mathbb{D}; \mathbb{C})$. Their system theoretic properties are identical (see [4, Exercise 2.5]). For every $\theta \in [0, 2\pi]$ the system (6) can be written as

$$\begin{aligned} \dot{\breve{z}}(e^{j\theta}, t) &= \breve{A}(e^{j\theta})\breve{z}(e^{j\theta}, t) + \breve{B}(e^{j\theta})\breve{u}(e^{j\theta}, t) \\ \breve{y}(e^{j\theta}, t) &= \breve{C}(e^{j\theta})\breve{z}(e^{j\theta}, t), \ t \ge 0. \end{aligned} \tag{7}$$

In Section 2 we analyze the LQR control problem for two examples and show that both the growth bounds and the transient bounds of the closed-loop operators for the finite and infinite string models are radically different. For one example the growth bounds satisfy $\omega_N < \omega_\infty$ and the transient factors increase without bound as $N \to \infty$, whereas for the other example $\omega_N > \omega_\infty$ and it has a transient bound of one. Sufficient conditions under which the solution to the LQR problem for the infinite string will serve as a useful paradigm for the long-but-finite strings are provided in Section 3. We also consider Riccati equations for the circulant matrix approximating system $\Sigma(\tilde{A}_N, \tilde{B}_N, \tilde{C}_N, 0)$, where $\tilde{A}_N$ is the circulant matrix approximant of the symbol $\breve{A}$ as defined in the Appendix. The Riccati solutions for the circulant approximating systems do exhibit very similar behavior to the infinite-dimensional ones as $N \to \infty$. In Section 4 we analyze yet another class of approximating Riccati equations which have been considered in the literature. They have similar convergence properties to the Toeplitz approximants. All results are illustrated by worked examples. Some Matlab simulations are presented in Section 5 and conclusions are drawn in Section 6. Notations and background results are collected in the Appendix.

2 Counterexamples

In this section we show that infinite string do not always capture the essence of the long-but-finite strings. We analyze two examples for which the growth bounds of the LQR closed-loop finite and infinite strings are significantly different. The first example illustrates the difference in stability between a finite and an infinite string.

Example 2.1 Consider the uncontrolled finite string model with real coefficients

$$\begin{aligned} \dot{z}_r(t) &= a_0 z_r(t) + a_1 z_{r-1}(t), \quad -N+1 \le r \le N \\ \dot{z}_{-N}(t) &= a_0 z_{-N}(t), \quad t \ge 0 \end{aligned}$$

with the system matrix

$$\mathbf{A}_N = \begin{bmatrix} a_0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ a_1 & a_0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & a_1 & a_0 & 0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & 0 & \cdots & a_1 & a_0 \end{bmatrix}.$$

$\mathbf{A}_N$ has the multiple eigenvalue a_0 and the growth bound

$$\omega_N = \max\{\mathrm{Re}(\lambda) : \lambda \in \sigma(\mathbf{A}_N)\} = a_0.$$

However the transient behavior depends strongly on N. We make this explicit by decomposing $\mathbf{A}_N = a_0 I_N + a_1 F_N$, where I_N is the $(2N+1)$ identity matrix and F_N is the $(2N+1)\times(2N+1)$ nilpotent matrix with $F_N^{2N+1}=0$. This gives

$$e^{\mathbf{A}_N t} = e^{a_0 t}\left(I_N + a_1 t F_N + \dots + \frac{1}{(2N)!}(a_1 t)^{2N} F_N^{2N}\right). \quad (8)$$

Noting that $\|F_N\|=1$ we can obtain the estimates $\|e^{\mathbf{A}_N t}\| \le t^{2N} e^{a_0 t} e^{|a_1|}$ for $t\ge 1$ and for $\varepsilon > 0$

$$\begin{aligned}\|e^{\mathbf{A}_N t}\| &\le e^{(a_0+|a_1|)t},\ t\ge 0;\\ \|e^{\mathbf{A}_N t}\| &\le M_\varepsilon(N) e^{(a_0+\varepsilon)t},\ t\ge 0.\end{aligned}$$

We now compare this with the infinite string model

$$\dot z_r(t) \;=\; a_0 z_r(t) + a_1 z_{r-1}(t),\ r\in\mathbb{Z},\ \ t\ge 0,$$

which is isomorphic to the system

$$\dot{\check z}(t) \;=\; (a_0 + a_1 e^{-\mathrm{j}\theta})\check z(t),\ \ t\ge 0,\ \ \theta\in[0,2\pi].$$

The system matrix $\check A(e^{\mathrm{j}\theta}) = a_0 + a_1 e^{-\mathrm{j}\theta}$ has the continuous spectrum $\sigma(\check A) = \{\lambda = x + \mathrm{j}y \mid (x-a_0)^2 + y^2 = a_1^2\}$. Thus

$$\|e^{At}\| = \|e^{\check A t}\|_\infty = \max_{\theta\in[0,2\pi]} |e^{(a_0+a_1 e^{-\mathrm{j}\theta})t}| = e^{(a_0+|a_1|)t},$$

and the growth bound $\omega_\infty = \sup\{\mathrm{Re}(\lambda) : \lambda\in\sigma(A)\} = a_0 + |a_1|$, which is larger than $\omega_N = a_0$. For a fixed positive $\varepsilon < |a_1|$ any transient bound $M_\varepsilon(N)$ increases as $N\to\infty$, whereas the transient bound of the infinite string is one. Clearly the finite and infinite strings exhibit very different stability behavior especially in the case $a_0 < 0$ and $a_0 + |a_1| > 0$. If $a_0 + |a_1| < 0$ this example can serve as a (trivial) LQR example with A and $\mathbf{A}_N$ representing the closed-loop operators.

The above example emphasizes that when comparing the behavior of finite and infinite string models both the growth bound and the transient factor are important indicators. In the following LQR example the transient factors are both 1, but the growth bound of the finite string is larger than that of the infinite string model.

Example 2.2 Let $\beta > 1$ be given. Consider the following finite string of the form (1)

$$\begin{aligned}\dot z_r(t) &= z_r(t) + u_r(t) + \beta u_{r-1}(t),\ -N+1\le r\le N\\ \dot z_{-N}(t) &= z_{-N}(t) + u_{-N}(t),\\ y_r(t) &= z_r(t),\ -N\le r\le N,\ t\ge 0.\end{aligned}$$

which can be written in the compact form (2) with $\mathbf{A}_N = \mathbf{C}_N = I_N$ and $\mathbf{B}_N = \begin{bmatrix} 1 & 0 & 0 & \dots & 0\\ \beta & 1 & 0 & \dots & 0\\ \dots & \dots & \dots & \dots & \dots\\ 0 & 0 & \dots & \beta & 1\end{bmatrix}$. The finite string is obviously stabilizable and detectable for all N. Factorize $\mathbf{B}_N\mathbf{B}_N^* = W_N \mathrm{diag}(\beta_k(N)) W_N^*$, where W_N is a $(2N+1)\times(2N+1)$ unitary matrix. Then the solution Q_N to the corresponding control Riccati equation is readily calculated $Q_N = W_N \mathrm{diag}\left(\frac{1+\sqrt{1+\beta_k(N)}}{\beta_k(N)}\right) W_N^*$. Hence $\|Q_N\| = \max_{k=0,\dots,2N} \dfrac{1+\sqrt{1+\beta_k(N)}}{\beta_k(N)}$, which is achieved at $\beta_{\min}(N)$, the minimum value of $\beta_k(N)$. The closed-loop operator is given by

$$\mathbf{A}_N - \mathbf{B}_N(\mathbf{B}_N)^* Q_N = W_N \mathrm{diag}\left(-\sqrt{1+\beta_k(N)}\right) W_N^*.$$

Hence $\|e^{(\mathbf{A}_N - \mathbf{B}_N(\mathbf{B}_N)^* Q_N)t}\| = e^{-\sqrt{1+\beta_{\min}(N)}t}$. We claim that for $\beta > 1$ one eigenvalue of $\mathbf{B}_N(\mathbf{B}_N)^*$ approaches 0 as $N\to\infty$. It is readily verified that $\mathbf{B}_N\mathbf{B}_N^* v_N = w_N$, where

$$\begin{aligned} v_N &= (-\beta^{-1}, \beta^{-2}, -\beta^{-3}, \dots, \beta^{-2N}, -\beta^{-2N-1})^T,\\ w_N &= (0,0,0,\dots,0,-\beta^{-2N-1})^T.\end{aligned}$$

Since $\beta > 1$, one eigenvalue must become arbitrarily small as $N\to\infty$ which means that $\|Q_N\|\to\infty$, and one eigenvalue of $\mathbf{A}_N - \mathbf{B}_N\mathbf{B}_N^* Q_N$ approaches -1 as $N\to\infty$. Hence the growth bound $\omega_N \to -1$ as $N\to\infty$ (see also Lemma A.2 *(2)*).
We show below that this behavior is very different from that of the infinite string

$$\begin{aligned}\dot z_r(t) &= z_r(t) + u_r(t) + \beta u_{r-1}(t),\\ y_r(t) &= z_r(t),\quad r\in\mathbb{Z},\quad t\ge 0.\end{aligned}$$

This system is isomorphic via Fourier transforms to

$$\begin{aligned}\dot{\check z}(t) &= \check z(t) + (1+\beta e^{-\mathrm{j}\theta})\check u(t),\\ \check y(t) &= \check z(t),\ \ t\ge 0,\ \ \theta\in[0,2\pi].\end{aligned}$$

It is clearly exponentially detectable and it is exponentially stabilizable, since the matrix $[\lambda - 1 : 1+\beta e^{-\mathrm{j}\theta}]$ has rank one for all $\lambda\in\mathbb{C}$, $\mathrm{Re}(\lambda)\ge 0$ and all $\theta\in[0,2\pi]$ (see [5,6]). The LQR Riccati equation

$$\check Q(e^{\mathrm{j}\theta}) + \check Q(e^{\mathrm{j}\theta}) - \check Q(e^{\mathrm{j}\theta})(1+\beta e^{-\mathrm{j}\theta})(1+\beta e^{-\mathrm{j}\theta})^* \check Q(e^{\mathrm{j}\theta}) + 1 = 0$$

has the unique positive solution $\check Q(e^{\mathrm{j}\theta}) = \frac{1+\sqrt{2+\beta^2+2\beta\cos\theta}}{1+\beta^2+2\beta\cos\theta}$ with norm $\|\check Q\|_\infty = \frac{1+\sqrt{1+(1-\beta)^2}}{(1-\beta)^2}$. The closed-loop operator $\check A_Q = \check A - \check B\check B^*\check Q$ is $\check A_Q(e^{\mathrm{j}\theta}) = -\sqrt{2+\beta^2+2\beta\cos\theta}$, $\theta\in[0,2\pi]$. Hence its growth bound $\omega_\infty = -\sqrt{2+\beta^2-2\beta} < -1$ and its transient factor is 1. Notice that ω_∞ decreases as β

increases. In contrast, for the finite string the growth bound satisfies $\omega_N \to -1$ as $N \to \infty$ for all $\beta > 1$.

So for two examples we have shown that both the growth bounds and the transient factors can be radically different. The obvious conclusion is that the infinite-dimensional string is not always a useful paradigm for understanding the behavior of long-but-finite strings.

3 Main Results

In this section we give conditions under which the solution to the LQR problem for the infinite string will serve as a useful paradigm for the long-but-finite strings. We use the notation and assumptions from Section 2.
The standard result on Riccati equations Curtain and Zwart [4, Theorem 6.2.7] and the result on exponential stabilizability and detectability Curtain *et al* [5,6, Theorems 4.1,4.2] yield the following result for the infinite-dimensional system $\Sigma(A,B,C,0)$ defined in (5).

Theorem 3.1 *The system $\Sigma(A,B,C,0)$ is exponentially stabilizable (detectable) if and only if $(\check{A}(e^{j\theta}),\check{B}(e^{j\theta}),\check{C}(e^{j\theta}),0)$ is stabilizable (detectable) for each $\theta \in [0,2\pi]$. If the above holds, then the control Riccati equation*

$$A^*Q + QA - QBB^*Q + C^*C = 0 \tag{9}$$

has a unique nonnegative solution Q and $A_Q = A - BB^*Q$ generates an exponentially stable semigroup. Moreover, the control Riccati equation

$$\check{A}^*\check{Q} + \check{Q}\check{A} - \check{Q}\check{B}\check{B}^*\check{Q} + \check{C}^*\check{C} = 0 \tag{10}$$

has a unique nonnegative solution $\check{Q} \in \mathbf{L}_\infty(\partial\mathbb{D};\mathbb{C})$ and $\check{A}_Q = \check{A} - \check{B}\check{B}^*\check{Q}$ generates an exponentially stable semigroup. Furthermore, $\check{Q}(e^{j\theta})$ is continuous in θ on $[0,2\pi]$.

The continuity property follows from Lancaster and Rodman [15, Theorem 11.2.1].
The problem of approximating solutions to operator Riccati equations has received much attention in the literature. However, the strongest convergence results (see Ito [11]) are achieved only if the input and output spaces are finite-dimensional, which is never the case for spatially invariant systems. However, we can apply the theory in Kappel and Salamon [13] applied to (5) with (2) as a sequence of approximating control systems.
Denote by $\pi^N : Z = l_2(\mathbb{Z},\mathbb{C}) \to \mathbb{C}^{2N+1}$ the natural projection with $i^N : \mathbb{C}^{2N+1} \to l_2(\mathbb{Z},\mathbb{C})$ the corresponding injection map: $\pi^N i^N = I_{2N+1}$. Denote $Z^N := \mathbb{C}^{2N+1}$ with the induced inner product $\langle x,y\rangle_N = \langle i^N x, i^N y\rangle_{l_2}$. Then $\mathbf{A}_N,\mathbf{B}_N,\mathbf{C}_N$ are Toeplitz matrix representations of the maps $\pi^N A|_{Z^N}$, $\pi^N B|_{Z^N}$, $\pi^N C|_{Z^N}$, with Z^N as the state space, input space and output space. For simplicity of notation we use the same notation for the maps as for the matrices, and we call $\Sigma(\mathbf{A}_N,\mathbf{B}_N,\mathbf{C}_N,0)$ the Toeplitz approximating systems for the infinite-dimensional string $\Sigma(A,B,C,0)$. Moreover, by expressions like "$\mathbf{A}_N$ converges strongly to A as $N \to \infty$" we shall mean the more precise $\lim_{N\to\infty} \|i^N \mathbf{A}_N \pi^N z - Az\| = 0$ for all $z \in l_2(\mathbb{Z},\mathbb{C})$. We say that the Toeplitz approximating systems $\Sigma(\mathbf{A}_N,\mathbf{B}_N,\mathbf{C}_N,0)$ *converge strongly* to the infinite-dimensional string $\Sigma(A,B,C,0)$ if as $N \to \infty$, $\mathbf{A}_N,\mathbf{B}_N,\mathbf{C}_N,\mathbf{A}_N^*,\mathbf{B}_N^*,\mathbf{C}_N^*$ converge strongly to the respective operators A,B,C,A^*,B^*,C^*. We also need following properties.

Definition 3.2 *The Toeplitz approximating systems $\Sigma(\boldsymbol{A}_N,\boldsymbol{B}_N,\boldsymbol{C}_N,0)$ are* uniformly stabilizable and detectable *if there exist $\boldsymbol{F}_N,\boldsymbol{L}_N \in \mathbb{C}^{(2N+1)\times(2N+1)}$, $F,L \in \mathcal{L}(l_2(\mathbb{Z},\mathbb{C}))$ such that $\boldsymbol{F}_N,\boldsymbol{L}_N,\boldsymbol{F}_N^*,\boldsymbol{L}_N^*$ converge strongly as $N \to \infty$ to the respective operators F,L,F^*,L^*, and there exist constants $M \geq 1, \beta > 0$ such that for sufficiently large $N \in \mathbb{N}$, $\|e^{(\boldsymbol{A}_N+\boldsymbol{B}_N\boldsymbol{F}_N)t}\| \leq Me^{-\beta t}$, $\|e^{(\boldsymbol{A}_N+\boldsymbol{L}_N\boldsymbol{C}_N)t}\| \leq Me^{-\beta t}$, $t \geq 0$.*

An application of Kappel and Salamon [13, Theorem 1, Proposition 1] and Ito [11] yields the following theorem.

Theorem 3.3 *Suppose that $\Sigma(A,B,C,0)$ is exponentially stabilizable and detectable and the sequence of the Toeplitz approximating systems $\Sigma(\boldsymbol{A}_N,\boldsymbol{B}_N,\boldsymbol{C}_N,0)$ is uniformly stabilizable and detectable and converges strongly to $\Sigma(A,B,C,0)$. For the state linear systems (5) and (2) let $Q \in \mathcal{L}(l_2(\mathbb{Z},\mathbb{C}))$ and $Q_N \in \mathcal{L}(Z^N)$ denote the unique nonnegative solutions of their respective Riccati equations (9) and*

$$\boldsymbol{A}_N^* Q_N + Q_N\boldsymbol{A}_N - Q_N\boldsymbol{B}_N\boldsymbol{B}_N^*Q_N + \boldsymbol{C}_N^*\boldsymbol{C}_N = 0. \tag{11}$$

Then Q_N converges strongly to Q and so $\|Q_N\|$ is uniformly bounded in N. Denote $A_Q := A - BB^*Q$ and $A_{Q_N} := \boldsymbol{A}_N - \boldsymbol{B}_N\boldsymbol{B}_N^*Q_N$. Then as $N \to \infty$, A_{Q_N} converges strongly to A_Q and $e^{A_{Q_N}t}$ converges strongly to e^{A_Qt} uniformly on compact time intervals.
Moreover, there exist positive constants $\overline{M},\mu$ such that

$$\|e^{A_Qt}\| \leq \overline{M}e^{-\mu t}, \quad \|e^{A_{Q_N}t}\| \leq \overline{M}e^{-\mu t} \text{ for all } t \geq 0, \tag{12}$$

and for all $u \in \ell_2$ as $N \to \infty$, we have
$\|C(\cdot I - A_Q)^{-1}Bu - i^N\boldsymbol{C}_N(\cdot I_N - A_N)^{-1}\boldsymbol{B}_N\pi^N u\|_{\mathbf{H}_2} \to 0,$
$\|B^*Q(\cdot I - A_Q)^{-1}Bu - i^N\boldsymbol{B}_N^*Q_N(\cdot I_N - A_N)^{-1}\boldsymbol{B}_N\pi^N u\|_{\mathbf{H}_2} \to 0.$

Note that the counterexample (4.1) in [13] shows that in general it is not true that
$\|C(\cdot I - A_Q)^{-1}Bu - i^N\mathbf{C}_N(\cdot I_N - A_{Q_N})^{-1}\mathbf{B}_N\pi_N u\|_{\mathbf{H}_\infty} \to 0$
as $N \to \infty$. We also remark that the solutions Q_N of (11) are not Toeplitz in general.

Sufficient conditions for uniform stabilizability and detectability are provided in the following propositions.

Proposition 3.4 *The Toeplitz approximating systems $\Sigma(\boldsymbol{A}_N,\boldsymbol{B}_N,\boldsymbol{C}_N,0)$ are uniformly detectable if any of the following equivalent statements is satisfied:*

(1) The Toeplitz operator $T(\check{C})$ (see (A.1)) is invertible.

(2) *The Toeplitz operator $T(\check{C})$ is a Fredholm operator of index zero.*
(3) *$\check{C}$ has no zeros on $\partial\mathbb{D}$ and* $\mathrm{wind}(\check{C},0)=0$. *Moreover the stability margin can be made arbitrarily large.*

Proposition 3.5 *The Toeplitz approximating systems $\Sigma(\boldsymbol{A}_N,\boldsymbol{B}_N,\boldsymbol{C}_N,0)$ are uniformly stabilizable if any of the following equivalent statements is satisfied:*

(1) *The Toeplitz operator $T(\check{B})$ is invertible.*
(2) *The Toeplitz operator $T(\check{B})$ is a Fredholm operator of index zero.*
(3) *$\check{B}$ has no zeros on $\partial\mathbb{D}$ and* $\mathrm{wind}(\check{B},0)=0$. *Moreover the stability margin can be made arbitrarily large.*

By duality it suffices to prove Proposition 3.4.
Proof: The equivalence *(1) ⇔ (2) ⇔ (3)* follows from [3] (Theorem 1.10, Theorem 1.17, $\mathrm{ind}(T(\check{C}))=-\mathrm{wind}(\check{C},0)$, $\check{C}$ is continuous).
By Lemma A.2.(2), $T(\check{C})$ is invertible if and only if there exists a nonzero γ such that for sufficiently large N, we have $\lambda_{\min}(\mathbf{C}_N^*\mathbf{C}_N)\geq\gamma^2$. Note that the condition is equivalent to

$$\langle\mathbf{C}_N z_N,\mathbf{C}_N z_N\rangle\geq\gamma^2\|z_N\|^2 \quad \text{for all} \quad z_N\in Z^N.$$

We show uniform detectability by using $\mathbf{L}_N=-\alpha^2\mathbf{C}_N^*$. Then by [10, Lemma 5.5.11] we have that

$$\|e^{(\mathbf{A}_N-\alpha^2\mathbf{C}_N^*\mathbf{C}_N)t}\|\leq e^{\lambda_{\max}(\mathbf{A}_N+\mathbf{A}_N^*-2\alpha^2\mathbf{C}_N^*\mathbf{C}_N)t/2}.$$

Now for $z_N\in Z^N$ we calculate

$$\begin{aligned}\langle(\mathbf{A}_N+\mathbf{A}_N^*-2\alpha^2\mathbf{C}_N^*\mathbf{C}_N)z_N,z_N\rangle &\leq 2(\|\mathbf{A}_N\|-\alpha^2\gamma^2)\|z_N\|^2\\ &\leq -2\beta\|z_N\|^2\end{aligned}$$

for arbitrarily large β by choosing α sufficiently large. Hence $\|e^{(\mathbf{A}_N-\alpha^2\mathbf{C}_N^*\mathbf{C}_N)t}\|\leq e^{-\beta t}$. ■

Strong convergence is insufficient to draw conclusions about the spectrum of A_{Q_N}. However, we note that in our Example 2.2 we only have uniform stabilizability if $\beta<1$. In this case we do not even have strong convergence. If only one of $\check{A},\check{B},\check{C}$ depends on θ, then we can prove better convergence results.

Proposition 3.6 *Consider $\check{A}=a_0,\check{B}=b_0$, $\check{C}\in\mathbf{H}_\infty$ and assume that the conditions in Theorem 3.3 are satisfied. Then there holds*

(1) $\limsup_{N\to\infty}\|Q_N\|=\|Q\|$ *and* $\|e^{A_{Q_N}t}\|\leq e^{-|\mathrm{Re}(a_0)|t}$ *for all* $t\geq0$.
(2) *For all $t\geq0$ there holds $\|e^{A_{Q_N}t}\|\leq e^{\omega_N t}$, where ω_N is the growth bound of $e^{A_{Q_N}t}$.*
(3) *$\omega_N\to\omega_\infty$ as $N\to\infty$.*

Proof: *(1)*: First we note that the unique solution to the infinite-dimensional Riccati equation is given by

$$\check{Q}(e^{j\theta})=\left(\mathrm{Re}(a_0)+\sqrt{(\mathrm{Re}(a_0))^2+|b_0|^2|\check{C}(e^{j\theta})|^2}\right)/|b_0|^2,$$

where $\mathrm{Re}(\cdot)$ denotes the real part of a complex number. The growth bound of the corresponding closed-loop operator is

$$\omega_\infty=-\sqrt{(\mathrm{Re}(a_0))^2+|b_0|^2\min_{\theta\in[0,2\pi]}|\check{C}(e^{j\theta})|^2}.$$

Using the factorization $\mathbf{C}_N^*\mathbf{C}_N=V_N\mathrm{diag}\{\gamma_k^2(N)\}V_N^*$, where V_N is unitary, we obtain the unique solution of (11) to be

$$Q_N=V_N\mathrm{diag}\left(\frac{\mathrm{Re}(a_0)+\sqrt{(\mathrm{Re}(a_0))^2+|b_0|^2\gamma_k^2(N)}}{|b_0|^2}\right)V_N^*,$$

$$A_{Q_N}=V_N\mathrm{diag}\left(-\sqrt{(\mathrm{Re}(a_0))^2+|b_0|^2\gamma_k^2(N)}\right)V_N^*.$$

Notice that

$$\|Q_N\|=\frac{\mathrm{Re}(a_0)+\sqrt{(\mathrm{Re}(a_0))^2+|b_0|^2\max_{k=0,..2N}\{\gamma_k^2(N)\}}}{|b_0|^2}.$$

So appealing to Lemma A.2 (1) we have $\limsup_{N\to\infty}\|Q_N\|=\|\check{Q}\|_\infty(=\|Q\|)$. The growth bound of $e^{A_{Q_N}t}$ is

$$\omega_N=-\sqrt{(\mathrm{Re}(a_0))^2+|b_0|^2\min_{k=0,\dots,2N}\{\gamma_k^2(N)\}}, \tag{13}$$

and

$$\|e^{A_{Q_N}t}\|\leq e^{-t\sqrt{(\mathrm{Re}(a_0))^2+|b_0|^2\min_{k=0,\dots,2N}\{\gamma_k^2(N)\}}}\leq e^{-|\mathrm{Re}(a_0)|t},$$

and this is independent of uniform detectability.
(2): This follows since A_{Q_N} is self-adjoint.
(3): We consider first the case when $T(\check{C})$ is not invertible. Then $\min_{\theta\in[0,2\pi]}|\check{C}(e^{j\theta})|$ and $\omega_\infty=-|\mathrm{Re}(a_0)|$. From Lemma A.2 (2) we have that $\lim_{N\to\infty}\min_{k=0,\dots,2N}\{\gamma_k(N)\}=0$. Then ω_N from (13) converges to ω_∞ as $N\to\infty$.
Assume now that $T(\check{C})$ is invertible. The condition $\check{C}\in\mathbf{H}_\infty$ means that the corresponding Toeplitz operator is lower triangular, and so $T^{-1}(\check{C})=T(\check{C}^{-1})$ (see [3, Proposition 1.13]). Using now Lemma A.2 (5) we obtain $\lim_{N\to\infty}\min_{k=0,\dots,2N}\{\gamma_k(N)\}=\frac{1}{\|T^{-1}(\check{C})\|}=\frac{1}{\|T(\check{C}^{-1})\|}=\min_{\theta\in[0,2\pi]}|\check{C}(e^{j\theta})|$, where $\gamma_k(N)$ are the singular values of $\mathbf{C}_N$. Hence $\omega_N\to\omega_\infty$ as $N\to\infty$.

Proposition 3.7 *Consider $\check{A} = a_0, \check{C} = c_0$ and assume that the conditions in Theorem 3.3 are satisfied.*

(1) If $Re(a_0) > 0$, then $\Sigma(\mathbf{A}_N, \mathbf{B}_N, \mathbf{C}_N, 0)$ is uniformly stabilizable if and only if $T(\check{B})$ is invertible.
(2) For all $t \geq 0$ there holds $\|e^{A_{Q_N}t}\| \leq e^{\omega_N t}$, where ω_N is the growth bound of $e^{A_{Q_N}t}$.
(3) If $T(\check{B})$ is not invertible, then $\omega_N \to -|\mathrm{Re}(a_0)|$ as $N \to \infty$.
(4) If $T(\check{B})$ is invertible and $\check{B} \in \mathbf{H}_\infty$, then $\omega_N \to \omega_\infty$ as $N \to \infty$.

Proof: *(1)*: For the sufficiency see Proposition 3.5. We prove necessity by contradiction. Assume that $T(\check{B})$ is not invertible. From Lemma A.2 (2) we have that

$$\lim_{N\to\infty} \min_{k=0,\ldots,2N} \{\beta_k(N)\} = 0.$$

where $\beta_k(N)$ are the singular values of $\mathbf{B}_N$. We can factorize $\mathbf{B}_N\mathbf{B}_N^* = V_N \mathrm{diag}\{\beta_k^2(N)\} V_N^*$, where V_N is unitary, to obtain the unique solution of (11) to be

$$Q_N = V_N \mathrm{diag}\left(\frac{\mathrm{Re}(a_0) + \sqrt{(\mathrm{Re}(a_0))^2 + |c_0|^2\beta_k^2(N)}}{\beta_k(N)^2}\right) V_N^*.$$

Thus $\|Q_N\| \to \infty$ and Q_N cannot converge strongly to Q. So, by Theorem 3.3, $\Sigma(\mathbf{A}_N, \mathbf{B}_N, \mathbf{C}_N, 0)$ cannot be uniformly stabilizable.
The statements *(2),(3)* and *(4)* follow as in the proof of Proposition 3.6, parts *(2)* and *(3)*. ■

Proposition 3.7 explains Example 2.2, since the $T(\check{B})$ is invertible if and only if $\beta < 1$. We also remark that the apparent contradiction between parts *(1)* and *(3)* lies in the fact that Definition 3.2 requires a bounded feedback gain $\|F_N\|$, whereas $\|\mathbf{B}_N Q_N\| \to \infty$ as $N \to \infty$.

The discretization of partial differential equations leads to systems with a real $\check{A}$ operator and constant $\check{B}, \check{C}$ operators (see El-Sayed and Krishnaprasad [7]). For such systems we also obtain nice convergence results (see also Section 5, Example 5.1 case 2).

Proposition 3.8 *Suppose that $\check{A} = \check{A}^*$, $\check{B} = b_0 \neq 0$, $\check{C} = c_0 \neq 0$. Then $\lim_{N\to\infty} \|Q_N\| = \|Q\|$ and the growth bound of $e^{A_{Q_N}t}$ converges to ω_∞ with $\|e^{A_{Q_N}t}\| \leq e^{\omega_\infty t}$.*

Proof: Note that the growth bound of the infinite-dimensional system is

$$\omega_\infty = -\sqrt{\min_{\theta\in[0,2\pi]} |\check{A}(e^{j\theta})|^2 + |c_0|^2|b_0|^2}.$$

We diagonalize the self-adjoint $\mathbf{A}_N = U_N \mathrm{diag}(\alpha_k(N)) U_N^*$ and find the solution to (11) to be

$$Q_N = U_N \mathrm{diag}\left(\frac{\alpha_k(N) + \sqrt{\alpha_k^2(N) + |c_0|^2|b_0|^2}}{|b_0|^2}\right) U_N^*,$$

and so $\|Q_N\| = \max_{k=1,\ldots,2N}\left(\frac{\alpha_k(N) + \sqrt{\alpha_k^2(N) + |c_0|^2|b_0|^2}}{|b_0|^2}\right)$.
The closed-loop operator

$$A_{Q_N} = U_N \mathrm{diag}\left(-\sqrt{\alpha_k^2(N) + |c_0|^2|b_0|^2}\right) U_N^*$$

is self-adjoint. The rest of the proof is similar to that in Proposition 3.6 with the important difference that

$$\lim_{N\to\infty} \min_{k=0,\ldots,2N} \alpha_k(N) = \min_{\theta\in[0,2\pi]} \check{A}(e^{j\theta})$$

and $\lim_{N\to\infty} \max_{k=0,\ldots,2N} \alpha_k(N) = \max_{\theta\in[0,2\pi]} \check{A}(e^{j\theta})$, since $\mathbf{A}_N$ is a self-adjoint Toeplitz matrix (see Lemma A.1). ■

In our simulations we obtained convergence also for $\check{A} \neq \check{A}^*$ (see Section 5, Example 5.2, except for case 4 in which $T(\check{B})$ is not invertible). We conjecture that similar results to Proposition 3.8 and Proposition 3.7 can be obtained for the case $\check{A} \neq \check{A}^*$. We remark that when A is not self-adjoint the convergence rate of the growth bound is often slow (see Section 5, Example 5.2, cases 1,3 and 8).

Example 3.9 A spatial discretization of the bi-infinite heated rod

$$\frac{\partial z}{\partial t}(t,x) = \alpha\frac{\partial^2 z}{\partial x^2}(t,x) + u(t,x),\ x \in \mathbb{R},\ t \geq 0,\ \alpha \neq 0.$$

with $z_r(t) := z(t,r)$, $u_r(t) := u(t,r)$, $y_r(t) := z(t,r)$, $r \in \mathbb{Z}$, leads to the spatially invariant system

$$\dot{z}_r(t) = \alpha(z_{r-1}(t) - 2z_r(t) + z_{r+1}(t)) + u_r(t),\ r \in \mathbb{Z},\ t \geq 0,$$

The solution to its Riccati equation is given by

$$\check{Q}(e^{j\theta}) = 2\alpha(\cos\theta - 1) + \sqrt{4\alpha^2(1-\cos\theta)^2 + 1},\ \theta \in [0, 2\pi],$$

and the closed-loop operator

$$\check{A}_Q(e^{j\theta}) = -\sqrt{4\alpha^2(1-\cos\theta)^2 + 1},\ \theta \in [0, 2\pi],$$

has the growth bound of -1. The corresponding Toeplitz approximating system has the solutions $Q_N = V_N D_N V_N^*$, where

V_N is a unitary matrix,

$$D_N = \text{diag}\left(2\alpha(\tau_k(N)-1)+\sqrt{4\alpha^2(1-\tau_k(N))^2+1}\right),$$

and $\tau_k(N) = \cos\dfrac{(k+1)\pi}{2N+2}$, $k = 0,...,2N$. Moreover, the closed-loop operator is given by

$$A_{Q_N} = V_N \text{diag}\left(-\sqrt{4\alpha^2(1-\tau_k(N))^2+1}\right)V_N^*,$$

and the growth bound is

$$\omega_N = -\sqrt{4\alpha^2(1-\cos\frac{\pi}{2N+2})^2+1}.$$

We remark that ω_N converges to -1 as $N\to\infty$.

In order to gain more information about the spectra of the approximating systems we examine the related *circulant approximants* of $\check{A},\check{B},\check{C}$ of dimension $n = 2N+1$ denoted by $\tilde{A}_N,\tilde{B}_N,\tilde{C}_N$ (see (A.3) in Appendix A).

Theorem 3.10 *Consider the exponentially stabilizable and detectable system* $\Sigma(A,B,C,0)$ *with* Q *the unique self-adjoint solution to the Riccati equation (9)*

(1) The Riccati equation

$$\tilde{A}_N^*\tilde{Q}_N+\tilde{Q}_N\tilde{A}_N-\tilde{Q}_N\tilde{B}_N\tilde{B}_N^*\tilde{Q}_N+\tilde{C}_N^*\tilde{C}_N = 0 \quad (14)$$

has a unique self-adjoint stabilizing solution $\tilde{Q}_N$ *which is the circular approximant of* $\check{Q}$.

(2) $\limsup_{N\to\infty}\|\tilde{Q}_N\| = \|\check{Q}\|_\infty = \|Q\|$.

(3) The growth bound $\tilde{\omega}_N$ *of* $e^{\tilde{A}_{Q_N}t}$ *satisfies*

$$\tilde{\omega}_N \le \omega_\infty, \quad \limsup_{N\to\infty}\tilde{\omega}_N = \omega_\infty,$$

where $\omega_\infty = \sup\{\text{Re}(\lambda),\lambda\in\sigma(A-BB^*Q)\}$, *the growth bound of* e^{A_Qt}. *Moreover, for all nonzero* $N\in\mathbb{N}$ *we have*

$$\|e^{\tilde{A}_{Q_N}t}\| \le e^{\omega_\infty t},\ \forall t\ge 0 \text{ and}$$

$$\|(\lambda I-\tilde{A}_{Q_N})^{-1}\|_\infty \le \frac{1}{\text{Re}(\lambda)-\omega_\infty} \qquad \text{for } \text{Re}(\lambda)>\omega_\infty.$$

(4)

$$\limsup_{N\to\infty}\left\|\begin{bmatrix}\tilde{C}_N\\ \tilde{B}_N^*\tilde{Q}_N\end{bmatrix}(\lambda I_N-\tilde{A}_{Q_N})^{-1}\tilde{B}_N\right\| = \left\|\begin{bmatrix}C\\ B^*Q\end{bmatrix}(\lambda I-A_Q)^{-1}B\right\|$$

for all $\lambda\in\mathbb{C}$, $\text{Re}(\lambda)>\omega_\infty$.

Proof: *(1)*: Note first that we have explicit formulas for the circulant approximants of $\check{A},\check{B},\check{C}$, for example,

$$\tilde{A}_N = U_N\text{diag}\left(\check{A}(e^{j(\frac{2\pi k}{2N+1})}\right)U_N^*, \quad k = 0,...,2N.$$

Hence $(\tilde{A}_N,\tilde{B}_N)$ will be stabilizable if and only if

$$(\lambda-\check{A}(e^{j\theta}))\check{B}(e^{j\theta})\ne 0 \text{ for } \theta = \frac{2\pi k}{2N+1}, k = 0,...,2N,$$

and $(\forall)\lambda\in\mathbb{C}$, $\text{Re}(\lambda)\ge 0$. A similar statement holds for detectability and these are implied by the exponential stabilizability and detectability of $\Sigma(\check{A},\check{B},\check{C},0)$. Hence (14) has a unique solution for all N. Taking circulant approximants term by term in (10) and properties (P2), (P3) show that the circulant approximants of $\check{Q}$ satisfy (14) and so $\tilde{Q}_N$ is the unique self-adjoint stabilizing solution of (10).

(2): From the proof of part *(1)* we can also write

$$\tilde{Q}_N = U_N\text{diag}\left(\check{Q}(e^{j(\frac{2\pi k}{2N+1})}\right)U_N^*,\ k = 0,...,2N,$$

where U_N is a unitary matrix. Since $\tilde{Q}_N$ is nonnegative definite we have that $\limsup_{N\to\infty}\|\tilde{Q}_N\| = \limsup_{N\to\infty}\max_{k=0,...,2N}\check{Q}(e^{j(\frac{2\pi k}{2N+1})}) = \max_{\theta\in[0,2\pi]}\check{Q}(e^{j\theta}) = \|\check{Q}\|_\infty = \|Q\|$.

(3): Since $\tilde{Q}_N$ is a circulant matrix, so is $\tilde{A}_{Q_N}$. Arguing as in the proof of part *(1)* we obtain $e^{\tilde{A}_{Q_N}t} = U_N\text{diag}\left(e^{\lambda_k^{(N)}t}\right)U_N^*$ where $\lambda_k^{(N)} = \check{A}_Q(e^{j(\frac{2\pi k}{2N+1})})$, $k = 0,...,2N$. Hence $\tilde{\omega}_N = \max_{k=0,...,2N}\{\text{Re}(\lambda_k^{(N)})\}\le\omega_\infty$, $\limsup_{N\to\infty}\tilde{\omega}_N = \omega_\infty$, and $\|e^{\tilde{A}_{Q_N}t}\|\le e^{\omega_\infty t}$. Finally, we have $\|(\lambda I-\tilde{A}_{Q_N})^{-1}\|\le\int_0^\infty\|e^{\tilde{A}_{Q_N}t}e^{-\lambda t}\|\,\mathrm{d}t\le\int_0^\infty e^{\omega_\infty t}e^{-\text{Re}(\lambda)t}\,\mathrm{d}t = \frac{1}{\text{Re}(\lambda)-\omega_\infty}$, $\text{Re}(\lambda)>\omega_\infty$, which proves part *(3)*.

(4): Note that $\tilde{\omega}_N\le\omega_\infty$ and so, for $\text{Re}(\lambda)>\omega_\infty$, we have

$$\tilde{C}_N(\lambda I_N-\tilde{A}_{Q_N})^{-1}\tilde{B}_N = U_ND_NU_N^*,$$

where $D_N = \text{diag}\left\{(\check{C}(\lambda I-\check{A})^{-1}\check{B})(e^{j(\frac{2\pi k}{2N+1})})\right\}$, and similarly for the other transfer function. Consequently, the limit is obtained. ■

In [20, Section 3] Willems solved an LQR problem for a string model with diagonal $\tilde{A}_N,\tilde{B}_N$ matrices and a circulant $\tilde{C}_N$ matrix. Although this is attractive from a computational viewpoint, it seems hard to justify from a modeling viewpoint. It would assume some sort of coupling between the first and the last vehicle. As the following example (the circulant version of the string model from Example 2.2) illustrates, this is not always realistic.

Example 3.11 Consider the infinite string $\Sigma(A,B,C,0)$ from Example 2.2. Its circulant approximating system

$\Sigma(\tilde{A}_N, \tilde{B}_N, \tilde{C}_N, 0)$ has

$$\tilde{A}_N = \tilde{C}_N = I_N, \text{ and } \tilde{B}_N = \begin{bmatrix} 1 & 0 & 0 & \dots & \beta \\ \beta & 1 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \beta & 1 \end{bmatrix}$$

and corresponds to the (fictious) finite string

$$\begin{aligned} \dot{z}_r(t) &= z_r(t) + u_r(t) + \beta u_{r-1}(t), \ -N+1 \le r \le N \\ \dot{z}_{-N}(t) &= z_{-N}(t) + u_{-N}(t) + \beta u_N(t), \\ y_r(t) &= z_r(t), \ -N \le r \le N, \ t \ge 0. \end{aligned}$$

Using the properties of circulant matrices (see Appendix A)

$$\tilde{B}_N \tilde{B}_N^* = \begin{bmatrix} 1+\beta^2 & \beta & 0 & \dots & \beta \\ \beta & 1+\beta^2 & \beta & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ \beta & 0 & \dots & \beta & 1+\beta^2 \end{bmatrix} = U_N \mathrm{diag}(\mu_k(N)) U_N^*,$$

where the eigenvalues of $\tilde{B}_N \tilde{B}_N^*$ are $\mu_k(N) = 1 + \beta^2 + 2\beta\cos\frac{2k\pi}{2N+1}, k = 0,\dots,2N$ and the unitary matrix $U_N = \frac{1}{\sqrt{2N+1}}\left[e^{-\frac{2\pi j r s}{2N+1}}\right]_{r,s=0,\dots,2N}$. Hence we can derive the explicit solution to the corresponding circular Riccati equation $\tilde{Q}_N = U_N \mathrm{diag}\left(\frac{1+\sqrt{1+\mu_k(N)}}{\mu_k(N)}\right) U_N^*$. Then $\|\tilde{Q}_N\| = \max_{k=0,..,2N} \frac{1+\sqrt{1+\mu_k(N)}}{\mu_k(N)} = \frac{1+\sqrt{2+\beta^2-2\beta\cos\frac{\pi}{2N+1}}}{1+\beta^2-2\beta\cos\frac{\pi}{2N+1}}$. Notice that $\|\tilde{Q}_N\| \to \|Q\|$ as $N \to \infty$. The closed-loop operator is given by $(\tilde{A}_N)_{\tilde{Q}_N} = \tilde{A}_N - \tilde{B}_N \tilde{B}_N^* \tilde{Q}_N = U_N D_N U_N^*$, where $D_N = \mathrm{diag}\left(-\sqrt{2+\beta^2-2\beta\cos\frac{\pi}{2N+1}}\right)$. So the eigenvalues of the closed-loop circulant approximating system all lie in the spectrum of A_Q and the growth bounds of their semigroups converge to $\omega_\infty = -\sqrt{2+\beta^2-2\beta}$ as $N \to \infty$.

We now relate the solutions $\tilde{Q}_N$ of the circulant Riccati equation (14) to the solutions to (11). Note that we use $|\cdot|_N$ instead of $|\cdot|_{2N+1}$ for the weak norm defined in Appendix A.

Theorem 3.12 *Assume that $\Sigma(A, B, C, 0)$ is stabilizable and detectable and $\Sigma(\boldsymbol{A}_N, \boldsymbol{B}_N, \boldsymbol{C}_N, 0)$ is uniformly stabilizable and detectable. Then the following hold*

(1) $|Q_N - \tilde{Q}_N|_N \to 0$ *and* $|A_{Q_N} - \tilde{A}_{Q_N}|_N \to 0$ *as* $N \to \infty$.

(2) The closed-loop transfer functions

$$G_N^{cl}(\lambda) = \begin{bmatrix} \boldsymbol{C}_N \\ \boldsymbol{B}_N^* Q_N \end{bmatrix} (\lambda I_N - A_{Q_N})^{-1} \boldsymbol{B}_N$$

and

$$G^{cl}(\lambda) = \begin{bmatrix} C \\ B^* Q \end{bmatrix} (\lambda I - A_Q)^{-1} B$$

satisfy
$\||G^{cl}(\cdot) - G_N^{cl}(\cdot)|_N\|_{\mathbf{H}_\infty} \to 0$ *and*
$\||G^{cl}(\cdot) - G_N^{cl}(\cdot)|_N\|_{\mathbf{H}_2} \to 0.$

(3) $\lim_{N\to\infty} \frac{1}{2N+1}\sum_{k=1}^{2N} \lambda_k(Q_N)^2 = \frac{1}{2\pi}\int_0^{2\pi} \check{Q}(e^{j\theta})^2 \,d\theta.$

(4) $\frac{1}{2N+1}\sum_{k=1}^{2N} \lambda_k(A_{Q_N} + A_{Q_N}^*)^2 \to \frac{1}{2\pi}\int_0^{2\pi} (\check{A}_Q(e^{j\theta}) + \check{A}_Q(e^{j\theta})^*)^2 \,d\theta$ *for* $N \to \infty$.

Proof: *(1)*: a. First we compare the solutions of the control Riccati equations of two exponentially stabilizable and detectable linear systems $\Sigma(A_i, B_i, C_i, 0); i = 1,2$ with bounded generating operators on the same state space. Let $Q_i, i = 1,2$ denote the nonnegative solutions to the Riccati equations

$$A_i^* Q_i + Q_i A_i - Q_i B_i B_i^* Q_i + C_i^* C_i = 0, \quad i = 1,2.$$

Then with $\Delta Q =: Q_1 - Q_2$ we have

$$\begin{aligned} A_{Q_1}^* \Delta Q + \Delta Q A_{Q_2} &= C_2^* C_2 - C_1^* C_1 + (A_2^* - A_1^*) Q_2 \\ &+ Q_1(A_2 - A_1) + Q_1(B_1 B_1^* - B_2 B_2^*) Q_2 \end{aligned}$$

and so we obtain

$$\begin{aligned} \Delta Q = \int_0^\infty e^{A_{Q_1}^* t} [C_1^* C_1 - C_2^* C_2 + (A_1^* - A_2^*) Q_2 \\ + Q_1(A_1 - A_2) + Q_1(B_2 B_2^* - B_1 B_1^*) Q_2] e^{A_{Q_2} t} \,dt, \end{aligned} \qquad (15)$$

where $A_{Q_i} = A - BB^* Q_i$, $i = 1,2$.
(1): b. Now from (P8) the Toeplitz matrix approximants and the circulant matrix approximants of A, B, C converge in the $|\cdot|_N$-norm and so $|\tilde{A}_N - \mathbf{A}_N|_N \to 0$, $|\tilde{B}_N - \mathbf{B}_N|_N \to 0$ and $|\tilde{C}_N - \mathbf{C}_N|_N \to 0$, as $N \to \infty$. We show that $|\tilde{Q}_N - Q_N|_N \to 0$ as $N \to \infty$ by taking estimates in (15) to obtain

$$\begin{aligned} |Q_N - \tilde{Q}_N|_N \le M_N(|(\mathbf{C}_N)^* \mathbf{C}_N - \tilde{C}_N^* \tilde{C}_N|_N + |\tilde{A}_N - \mathbf{A}_N|_N \\ (\|Q_N\| + \|\tilde{Q}_N\|) + \|\tilde{Q}_N\| \|Q_N\| |\mathbf{B}_N (\mathbf{B}_N)^* - \tilde{B}_N (\tilde{B}_N)^*|_N), \end{aligned}$$

where $M_N = \left(\int_0^\infty \|e^{\mathbf{A}_{Q_N} t}\|^2 \,dt \int_0^\infty \|e^{\tilde{A}_{Q_N} t}\|^2 \,dt\right)^{\frac{1}{2}}$. We have used property (P7) of the matrix norm from Appendix A. But both $\mathbf{L}_2$-norms are uniformly bounded in N (see (12) and Theorem 3.10 part *(3)*), as are $\|Q_N\|$ and $\|\tilde{Q}_N\|$ (see Theorem 3.3 and Theorem 3.10 part *(2)*). So there exists a positive constant γ such that

$$|Q_N - \tilde{Q}_N|_N \le \gamma(|\mathbf{C}_N - \tilde{C}_N|_N + |\mathbf{A}_N - \tilde{A}_N|_N + |\mathbf{B}_N - \tilde{B}_N|_N).$$

Hence $|\tilde{Q}_N - Q_N|_N \to 0$ and $|\tilde{A}_{Q_N} - A_{Q_N}|_N \to 0$ as $N \to \infty$.
(2): To obtain the bounds on the norm of $e^{A_{Q_N} t}$ we first recall from the proof of part *(1)* that $|A_N - \tilde{A}_N|_N \to 0$ as $N \to \infty$, etc. Then, since $|Q_N - \tilde{Q}_N|_N \to 0$ as $N \to \infty$, using

the properties (P6), (P7) from Appendix A we obtain

$$|A_{Q_N} - \tilde{A}_{Q_N}|_N = |A_N - \tilde{A}_N - \mathbf{B}_N\mathbf{B}_N^* Q_N + \tilde{B}_N\tilde{B}_N^*\tilde{Q}_N|_N \to 0$$

as $N \to \infty$. Consider the perturbation formula

$$e^{A_{Q_N}t} = e^{\tilde{A}_{Q_N}t} + \int_0^t e^{\tilde{A}_{Q_N}(t-s)}[\tilde{A}_{Q_N} - A_{Q_N}]e^{A_{Q_N}s}\,\mathrm{d}s. \quad (16)$$

Using again (P6), (P7) together with part *(3)* of Theorem 3.10 and (12) we obtain estimates for sufficiently large N

$$\begin{aligned}|e^{A_{Q_N}t} - e^{\tilde{A}_{Q_N}t}|_N &\le |\tilde{A}_{Q_N} - A_{Q_N}|_N \left(\int_0^t e^{\omega_\infty(t-s)}\overline{M}e^{-\mu s}\mathrm{d}s\right)\\ &\le |\tilde{A}_{Q_N} - A_{Q_N}|_N K e^{-\delta t},\end{aligned}$$

where $K = \frac{2\overline{M}}{|\omega_\infty+\mu|}$ and $\delta > \min\{\mu, -\omega_\infty\}$. Hence we have

$$\begin{aligned}&|(\lambda I_N - A_{Q_N})^{-1} - (\lambda I_N - \tilde{A}_{Q_N})^{-1}|_N\\ &\le K|\tilde{A}_{Q_N} - A_{Q_N}|_N \int_0^\infty e^{-\delta t}e^{-\mathrm{Re}(\lambda)t}\,\mathrm{d}t\\ &\le |\tilde{A}_{Q_N} - A_{Q_N}|_N \frac{K}{\mathrm{Re}(\lambda)+\delta} \to 0 \text{ as } N \to \infty, \quad (17)\end{aligned}$$

for $\mathrm{Re}(\lambda) > -\delta$. Then, for $\mathrm{Re}(\lambda) > 0$,

$$\begin{aligned}&|(\lambda I_N - A_{Q_N})^{-1}\mathbf{B}_N - (\lambda I_N - \tilde{A}_{Q_N})^{-1})\tilde{B}_N|_N\\ &\le |((\lambda I_N - A_{Q_N})^{-1} - (\lambda I_N - \tilde{A}_{Q_N})^{-1})\mathbf{B}_N|_N\\ &+|(\lambda I_N - \tilde{A}_{Q_N})^{-1}(\mathbf{B}_N - \tilde{B}_N)|_N\\ &\le |(\lambda I_N - A_{Q_N})^{-1} - (\lambda I_N - \tilde{A}_{Q_N})^{-1}|_N\|\mathbf{B}_N\|\\ &+\frac{1}{|\omega_\infty|}|\mathbf{B}_N - \tilde{B}_N|_N \to 0 \text{ as } N \to \infty,\end{aligned}$$

where we have used part *(3)* of Theorem 3.10. Repeating this type of reasoning we obtain, for $\mathrm{Re}(\lambda) > -\delta$,

$$|\mathbf{C}_N(\lambda I_N - A_{Q_N})^{-1}\mathbf{B}_N - \tilde{C}_N(\lambda I_N - \tilde{A}_{Q_N})^{-1})\tilde{B}_N|_N \to 0$$

as $N \to \infty$. Now the estimate in (17) and the subsequent estimates are uniform in $\mathrm{Re}(\lambda) > 0$. Hence

$$\sup_{\mathrm{Re}(\lambda)\ge 0}|\mathbf{C}_N(\lambda I_N - A_{Q_N})^{-1}\mathbf{B}_N - \tilde{C}_N(\lambda I_N - \tilde{A}_{Q_N})^{-1})\tilde{B}_N|_N \to 0$$

as $N \to \infty$, and similarly for the other transfer function.
To establish the $\mathbf{H}_2$-norm convergence use (16) to obtain

$$\begin{aligned}&\int_0^\infty |e^{A_{Q_N}t} - e^{\tilde{A}_{Q_N}t}|_N^2\,\mathrm{d}t\\ &\le |A_{Q_N} - \tilde{A}_{Q_N}|_N^2 \left(\int_0^\infty \|e^{\tilde{A}_{Q_N}t}\|\mathrm{d}t\right)^2 \int_0^\infty \|e^{A_{Q_N}t}\|^2\mathrm{d}t\\ &\le |A_{Q_N} - \tilde{A}_{Q_N}|_N^2 \frac{1}{\omega_\infty^2}\frac{\overline{M}^2}{2\mu}.\end{aligned}$$

for sufficiently large N, where we have used Theorem 3.10 part *(3)* and (12).
Hence $\||(\cdot I - A_{Q_N})^{-1} - (\cdot I - \tilde{A}_{Q_N})^{-1}|_N\|_{\mathbf{H}_2} \to 0$ as $N \to \infty$. The rest follows as for the $\mathbf{H}_\infty$ result.
(3),(4): These follow from part *(2)* and Lemma A.3 in Appendix A. ∎

We remark that the convergence results for the transfer functions are necessarily weak. A simple calculation with the diagonal system with $\check{A} = a_0$, $\check{B} = b_0$, $\check{C} = c_0$ shows that we will never have $\|G^{cl} - i^N G_N^{cl}\pi^N\|_{\mathbf{H}_\infty} \to 0$, and $\|G^{cl} - i^N G_N^{cl}\pi^N\|_{\mathbf{H}_2} \to 0$. The most one could hope for is strong convergence $\|G^{cl}u - i^N G_N^{cl}\pi^N u\|_{\mathbf{H}_\infty} \to 0$, and $\|G^{cl}u - i^N G_N^{cl}\pi^N u\|_{\mathbf{H}_2} \to 0$ for all $u \in U$. While the strong convergence in the $\|\cdot\|_{\mathbf{H}_2}$-norm does hold (see Theorem 3.3, the $\|\cdot\|_{\mathbf{H}_\infty}$-norm convergence is unclear (see Counterexample 4.1 in [13]).
Theorem 3.12 gives a possible explanation for Example 2.2 where $\omega_N > \omega_\infty$ for $\beta > 1$; the system is not uniformly stabilizable. Similar gaps between ω_N and ω_∞ are found in numerical simulations (see Example 5.1 case 4 and Example 5.2 case 4). Example 2.1 shows that uniform stabilizability and detectability do not imply that $\limsup_{N\to\infty}\omega_N \to \omega_\infty$. For $a_1 \ge 0$, we have $\omega_N = a_0 < \omega_\infty = a_0 + a_1$. This difference is explained by the transient bounds $M(N)$ that increase drastically with N (see (8)). Similar results hold for $\mathbf{Q}_N$, the Toeplitz approximant of $\check{Q}$, which is not the same as Q_N.

Corollary 3.13 *Assume that $\Sigma(A,B,C,0)$ is stabilizable and detectable and the Toeplitz approximating systems $\Sigma(\boldsymbol{A}_N, \boldsymbol{B}_N, \boldsymbol{C}_N)$ are uniformly stabilizable and detectable for sufficiently large N. Then with $A_{\boldsymbol{Q}_N} := \boldsymbol{A}_N - \boldsymbol{B}_N\boldsymbol{B}_N^*\boldsymbol{Q}_N$ we have*

(1) $|\boldsymbol{Q}_N - \tilde{Q}_N|_N \to 0$ *and* $|A_{\boldsymbol{Q}_N} - \tilde{A}_{Q_N}|_N \to 0$ *as* $N \to \infty$.
(2) The closed-loop transfer functions

$$G_N^{cl}(\lambda) = \begin{bmatrix} \boldsymbol{C}_N \\ \boldsymbol{B}_N^*\boldsymbol{Q}_N \end{bmatrix} (\lambda I_N - A_{\boldsymbol{Q}_N})^{-1}\boldsymbol{B}_N \text{ and}$$

$$G^{cl}(\lambda) = \begin{bmatrix} C \\ B^*Q \end{bmatrix} (\lambda I - A_Q)^{-1}B$$

satisfy $\||G^{cl}(\cdot) - G_N^{cl}(\cdot)|_N\|_{\mathbf{H}_\infty} \to 0$ *and* $\||G^{cl}(\cdot) - G_N^{cl}(\cdot)|_N\|_{\mathbf{H}_2} \to 0$, *respectively.*

(3) $\lim_{N\to\infty} \frac{1}{2N+1}\sum_{k=1}^{2N} \lambda_k(\boldsymbol{Q}_N)^2 = \frac{1}{2\pi}\int_0^{2\pi} \check{Q}(e^{j\theta})^2\,\mathrm{d}\theta$ *and* $\lim_{N\to\infty}\|\boldsymbol{Q}_N\| = \|Q\|$.

(4) $\lim_{N\to\infty} \frac{1}{2N+1}\sum_{k=1}^{2N} \lambda_k(A_{\boldsymbol{Q}_N} + A_{\boldsymbol{Q}_N}^*)^2 = \frac{1}{2\pi}\int_0^{2\pi} (\check{A}_Q(e^{j\theta}) + \check{A}_Q(e^{j\theta})^*)^2\,\mathrm{d}\theta$.

Proof: *(1)*: This part follows from (P8) (see Appendix A).
(2),(3),(4): The proofs are analogous to those for Theorem 3.12. The extra result $\lim_{N\to\infty}\|\mathbf{Q}_N\| = \|Q\|$ follows from Lemma A.1, since $\mathbf{Q}_N$ is a self-adjoint Toeplitz operator. ∎

4 Alternative Toeplitz Approximants

Note that in [12] and other papers a different type of approximating Riccati equation was studied

$$\mathbf{A}_N^*\overline{Q}_N + \overline{Q}_N\mathbf{A}_N - \overline{Q}^N\mathbf{B}_N\mathbf{B}_N^*\overline{Q}_N + (\mathbf{C}^*\mathbf{C})_N = 0, \quad (18)$$

where $(\mathbf{C}^*\mathbf{C})_N$ is the matrix representation of the map $T^{(2N+1)}(C^*C) = \pi^N C^*C|_{Z^N}$ (recall that $\pi^N : \mathbb{Z} = \ell_2 \to \mathbb{C}^{2N+1} = Z^N$). These are easier to solve, since the term $(\mathbf{CC}^*)_N$ is a Toeplitz matrix, whereas $\mathbf{C}_N^*\mathbf{C}_N$ is not necessarily Toeplitz. They have similar convergence properties to (11).
First we show how the $(\mathbf{C}^*\mathbf{C})_N$ term can be reformulated to fit into the set-up of Theorem 3.3. Denote $\overline{C}_N = (\mathbf{C}^*\mathbf{C})_N^{\frac{1}{2}}$.

Proposition 4.1 *Suppose that $\Sigma(A,B,C,0)$ is exponentially stabilizable and detectable and that the Toeplitz approximating systems $\Sigma(\boldsymbol{A}_N,\boldsymbol{B}_N,(\boldsymbol{C}^*\boldsymbol{C})_N,0)$ are uniformly stabilizable and detectable. Let $Q \in \mathcal{L}(Z)$ and $\overline{Q}_N \in \mathcal{L}(Z^N)$ denote the nonnegative solutions of their respective Riccati equations (9) and (18). Then*

$$Qz = \lim_{N\to\infty} i^N\overline{Q}_N\pi^N z, \quad \forall z \in Z,$$

and $\|\overline{Q}_N\|$ is uniformly bounded in N. Furthermore, for sufficiently large N, $A_Q := A - BB^*Q$ and $\overline{A}_{Q_N} := \boldsymbol{A}_N - \boldsymbol{B}_N\boldsymbol{B}_N^*\overline{Q}_N$ generate exponentially stable semigroups with

$$\|e^{A_Q t}\| \le \overline{M}e^{-\mu t}, \quad \|e^{\overline{A}_{Q_N} t}\| \le \overline{M}e^{-\mu t} \text{ for all } t \ge 0,$$

and $i^N e^{\overline{A}_{Q_N} t} z \to e^{A_Q t}z, \quad \forall z \in Z$, uniformly on compact time intervals.

Proof: First we note that the Riccati equations can be associated with the systems $\Sigma(A,B,\sqrt{C^*C},0)$ and $\Sigma(\mathbf{A}_N,\mathbf{B}_N,\overline{C}_N,0)$, since $\overline{C}_N^*\overline{C}_N = (\mathbf{C}^*\mathbf{C})_N$. Next we show that these systems satisfy the assumptions of Theorem 3.3. $\Sigma(A,B,C^*C,0)$ is exponentially detectable if and only if $\Sigma(A,B,C,0)$ is. Hence $\Sigma(A,B,\sqrt{C^*C},0)$ is exponentially detectable (choose $\overline{L} = L(C^*C)^{\frac{1}{2}}$). Moreover, $\Sigma(\mathbf{A}_N,\mathbf{B}_N,\overline{C}_N,0)$ is uniformly stabilizable and detectable, since $\Sigma(\mathbf{A}_N,\mathbf{B}_N,(\mathbf{C}^*\mathbf{C})_N,0)$ is (choose $\overline{L}_N = \mathbf{L}_N(\mathbf{C}^*\mathbf{C})_N^{\frac{1}{2}}$).
It remains to show that $i^N\overline{C}_N|_{Z^N}z \to \sqrt{\mathbf{C}^*\mathbf{C}}z$ as $N \to \infty$. To ease notation, denote $R_N := (\mathbf{C}^*\mathbf{C})_N$, $\sqrt{R_N} := i^N\overline{C}_N\pi^N$ and $R := C^*C$. So the self-adjoint, nonnegative operators R_N, R satisfy $R_N z \to Rz$ as $N \to \infty$ and we need to show that $\sqrt{R_N}z \to \sqrt{R}z$ as $N \to \infty$. From [14, (3.45), p.282] we have the formula for the square root of a self-adjoint, nonnegative operator $\sqrt{R}z = \frac{1}{\pi}\int_0^\infty \frac{1}{\sqrt{\lambda}}(R+\lambda I)^{-1}Rz\,\mathrm{d}\lambda$. For a nonnegative operator P, we have that

$$\|(\lambda I + P)^{-1}\| \le \frac{1}{\lambda} \text{ and } \|P(\lambda I + P)^{-1}\| \le 2, \quad (19)$$

for $\lambda > 0$. For every fixed z and $\delta > 0$,

$$\begin{aligned}
&\|\frac{1}{\pi}\int_0^\delta \frac{1}{\sqrt{\lambda}}(P+\lambda I)^{-1}Pz\,\mathrm{d}\lambda\| \\
&\le \frac{1}{\pi}\frac{1}{\sqrt{\lambda}}\int_0^\delta \frac{1}{\sqrt{\lambda}}\|(I-\lambda(P+\lambda I)^{-1})z\|\,\mathrm{d}\lambda \\
&\le \frac{1}{\pi}\int_0^\delta \frac{1}{\sqrt{\lambda}}(1+\frac{\lambda}{\lambda})\|z\|\,\mathrm{d}\lambda \le \frac{4}{\pi}\|z\|\sqrt{\delta}.
\end{aligned}$$

Using the above inequality one can obtain

$$\begin{aligned}
&\|\sqrt{R_N}z - \sqrt{R}z\| \le \frac{8}{\pi}\|z\|\sqrt{\delta} \\
&+\frac{1}{\pi}\|\int_\delta^\infty \sqrt{\lambda}(R_N+\lambda I)^{-1}(R-R_N)(R+\lambda I)^{-1}zd\lambda\|.
\end{aligned}$$

Now, using (19), one can write

$$\begin{aligned}
&\|(\sqrt{R_N}-\sqrt{R})z\| \le \frac{8}{\pi}\|z\|\sqrt{\delta} \\
&+\frac{1}{\pi}\int_\delta^\infty \frac{1}{\sqrt{\lambda}}\|(R-R_N)(R+\lambda I)^{-1}z\|\,\mathrm{d}\lambda.
\end{aligned}$$

Since R_N converges strongly to R and using again (19), we also have that

$$\frac{1}{\sqrt{\lambda}}\|(R-R_N)(R+\lambda I)^{-1}z\| \le \frac{2\|R\|}{\lambda^{3/2}}$$

and with the Lebesgue convergence lemma we have

$$\frac{1}{\pi}\int_\delta^\infty \frac{1}{\sqrt{\lambda}}\|(R-R_N)(R+\lambda I)^{-1}z\|\,\mathrm{d}\lambda \to 0 \text{ as } N\to\infty.$$

Combining the last two results shows that $\sqrt{R_N}z \to \sqrt{R}z$ as $N \to \infty$, as required. ■

The following proposition gives conditions for the uniform detectability.

Proposition 4.2 *Under the notation of Lemma 4.1, the Toeplitz approximating systems $\Sigma(\boldsymbol{A}_N,\boldsymbol{B}_N,(\boldsymbol{C}^*\boldsymbol{C})_N,0)$ are uniformly detectable if $\min_{\theta\in[0,2\pi]}|\check{C}(e^{\mathrm{j}\theta})| > 0$. Moreover, the stability margin β in Definition 3.2 can be made as large as we please.*

Proof: Since $\min_{\theta\in[0,2\pi]}|\check{C}(e^{\mathrm{j}\theta})| > 0$, there exists a nonzero γ such that $\|\check{C}z\|^2 \ge \gamma^2\|z\|^2$ for $z \in \mathbf{L}_2(\partial\mathbb{D},\mathbb{C})$. Then we have

$$\begin{aligned}
&\langle\check{A}z,z\rangle + \langle z,\check{A}z\rangle - 2\alpha\langle\check{C}z,\check{C}z\rangle \\
&\le 2\|\check{A}\|\|z\|^2 - 2\alpha\gamma^2\|z\|^2 \le -\delta^2\|z\|^2
\end{aligned}$$

for sufficiently large and positive α. This implies that $\|e^{(A-\alpha C^*C)t}\| \le e^{-\delta^2 t}$, where δ can be made as large as we please. In particular, with $z_N = \pi^N z$, we obtain

$$\langle \mathbf{A}_N z_N, z_N\rangle + \langle z_N, \mathbf{A}_N z_N\rangle - 2\alpha\langle (\mathbf{C}^*\mathbf{C})_N z_N, z_N\rangle \leq -\delta^2 \|z_N\|^2$$

and $\|e^{(\mathbf{A}_N - \alpha(\mathbf{C}^*\mathbf{C})_N)t}\| \leq e^{-\delta^2 t}$. So the feedback $F_N = -\alpha \overline{C}_N$ gives uniform detectability. ∎

With Propositions 4.1 and 4.2, and the same proof as in Theorem 3.12 we obtain the following corollary.

Corollary 4.3 *Suppose that $\Sigma(A,B,C,0)$ is exponentially stabilizable and detectable and the Toeplitz approximating systems $\Sigma(\boldsymbol{A}_N, \boldsymbol{B}_N, (\boldsymbol{C}^*\boldsymbol{C})_N, 0)$ are uniformly stabilizable and detectable. Then the solutions $\overline{Q}_N$ and the closed-loop operator $\overline{A}_{Q_N} = \boldsymbol{A}_N - \boldsymbol{B}_N\boldsymbol{B}_N^*\overline{Q}_N$ to the Riccati equation (18) have the following properties.*

(1) $|\overline{Q}_N - \tilde{Q}_N|_N \to 0$, and $|\overline{A}_{Q_N} - \tilde{A}_{Q_N}|_N \to 0$ as $N \to 0$,
(2) The closed-loop transfer functions

$$\overline{G}_N^{cl}(\lambda) = \begin{bmatrix} \overline{C}_N \\ \boldsymbol{B}_N^*\overline{Q}_N \end{bmatrix} (\lambda I_N - \overline{A}_{Q_N})^{-1}\boldsymbol{B}_N$$

and $$G^{cl}(\lambda) = \begin{bmatrix} C \\ B^*Q \end{bmatrix} (\lambda I - A_Q)^{-1} B$$

satisfy $\||G^{cl}(\cdot) - \overline{G}_N^{cl}(\cdot)|_N\|_{\mathbf{H}_\infty} \to 0$, and $\||G^{cl}(\cdot) - \overline{G}_N^{cl}(\cdot)|_N\|_{\mathbf{H}_2} \to 0$, respectively.
(3) $\displaystyle\lim_{N\to\infty} \frac{1}{2N+1}\sum_{k=1}^{2N} \lambda_k(\overline{Q}_N)^2 = \frac{1}{2\pi}\int_0^{2\pi} \check{Q}(e^{j\theta})^2\, d\theta$.
(4) $\displaystyle\lim_{N\to\infty} \frac{1}{2N+1}\sum_{k=1}^{2N} \lambda_k(\overline{A}_{Q_N} + \overline{A}_{Q_N}^*)^2 = \frac{1}{2\pi}\int_0^{2\pi} (\check{A}_Q(e^{j\theta}) + \check{A}_Q(e^{j\theta})^*)^2\, d\theta$.

For the special case of only delays in C we obtain convergence results which are consistent with the claims in [12].

Proposition 4.4 *Consider $\check{A} = a_0, \check{B} = b_0 \neq 0$ and suppose that the assumptions in Lemma 4.1 are satisfied. Then there holds*

(1) $\lim_{N\to\infty} \|\overline{Q}_N\| = \|Q\|$.
(2) The growth bounds of $e^{\overline{A}_{Q_N}t}$ converge to those of $e^{A_Q t}$ and $\|e^{\overline{A}_{Q_N}t}\| \leq e^{\omega_N t} \leq e^{\overline{\omega}_\infty t}$.
(3) If $\displaystyle\min_{\theta\in[0,2\pi]} |\check{C}(e^{j\theta})| > 0$, then $\overline{Q}_N$ converges strongly to Q as $N \to \infty$.
(4) The feedback law $u_N = -\boldsymbol{B}_N\tilde{Q}_N z_N$ stabilizes the system with $\|e^{(\boldsymbol{A}_N - \boldsymbol{B}_N\boldsymbol{B}_N^*\tilde{Q}_N)t}\| \leq e^{\omega_\infty t}$.

Proof: *(1),(2)*: Similarly to the proof of Corollary 3.6 we can factorize $(C^*C)_N = V_N \mathrm{diag}(\gamma_k^2(N))V_N^*$ to obtain the unique solution

$$\overline{Q}_N = V_N \mathrm{diag}\left(\frac{\mathrm{Re}(a_0) + \sqrt{(\mathrm{Re}(a_0))^2 + |b_0|^2\gamma_k^2(N)}}{|b_0|^2}\right) V_N^*,$$

and $\overline{A}_{Q_N} = V_N \mathrm{diag}\left(-\sqrt{(\mathrm{Re}(a_0))^2 + b_0^2\gamma_k^2(N)}\right) V_N^*$. So the growth bound is

$$\omega_N = -\sqrt{(\mathrm{Re}(a_0))^2 + |b_0|^2 \min_{k=0,\ldots,2N}\{\gamma_k^2(N)\}},$$

and $\|e^{\overline{A}_{Q_N}t}\| \leq e^{-\omega_N t}$, independent of uniform detectability. From Lemma A.1, the maximum and minimum eigenvalues of $(C^*C)_N$ converge to those of $\|C\|^2$, which shows that $\|\overline{Q}_N\|$ converges to $\|Q\|$ and the growth bound of $e^{\overline{A}_{Q_N}t}$ converges to that of the infinite-dimensional system.
(3): If $\displaystyle\min_{\theta\in[0,2\pi]} |\check{C}(\theta)| > 0$, the strong convergence of $\overline{Q}_N$ follows from Proposition 4.1.
(4): Next we note that $\mathbf{A}_N - \mathbf{B}_N\mathbf{B}_N^*\tilde{Q}_N = \tilde{A}_{Q_N}$ and appeal to Theorem 3.10. Note that $\mathbf{A}_N = \tilde{A}_N$ and $\mathbf{B}_N = \tilde{B}_N$ since $\check{A}$ and $\check{B}$ are scalars. ∎

The above lemma shows that, whenever one has only delays in C, a good strategy is to use the feedback law $u_N = -\mathbf{B}_N\tilde{Q}_N z_N$, since $\tilde{Q}_N$ is easy to calculate. We illustrate this by an example.

Example 4.5 Consider the alternative Riccati equation (18) with $\mathbf{A}_N = \mathbf{B}_N = I_N$ and

$$(C^*C)_N = \begin{bmatrix} 1+\kappa^2 & \kappa & 0 \ldots & 0 \\ \kappa & 1+\kappa^2 & \kappa \ldots & 0 \\ \ldots & \ldots & \ldots \ldots & \ldots \\ 0 & 0 & \ldots\ \kappa & 1+\kappa^2 \end{bmatrix} \neq \mathbf{C}_N^*\mathbf{C}_N.$$

We write $(C^*C)_N = (1+\kappa^2)I_N + \kappa T_N$, where

$$T_N = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 1 & 0 & 1 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & 1 & 0 \end{bmatrix} = V_N \mathrm{diag}\left(2\cos\frac{(k+1)\pi}{2N+2}\right) V_N^*,$$

where V_N is a unitary matrix. Then the solution to (18) is $\overline{Q}_N = V_N \mathrm{diag}\left(1 + \sqrt{1+\rho_k(N)}\right) V_N^*$, $\rho_k(N) = 1 + \kappa^2 + 2\kappa\cos\frac{\pi}{2N+2}$. Hence $\|\overline{Q}_N\| = \displaystyle\max_{k=0,\ldots,2N}\left(1 + \sqrt{1+\rho_k(N)}\right) = 1 + \sqrt{2+\kappa^2+2\kappa\cos\frac{\pi}{2N+2}}$. The closed-loop operator is $\overline{A}_{Q_N} = \mathbf{A}_N - \mathbf{B}\mathbf{B}_N^*\overline{Q}_N = V_N \mathrm{diag}\left(-\sqrt{1+\rho_k(N)}\right) V_N^*$, and its growth bound is $-\sqrt{2+\kappa^2-2\kappa\cos\frac{\pi}{2N+2}}$.
For the corresponding infinite-dimensional problem we have $\check{Q}(e^{j\theta}) = \left(1 + \sqrt{2+\kappa^2+2\kappa\cos\theta}\right)$ with norm $\|\check{Q}\| = 1 + \sqrt{1+(1+\kappa)^2}$. The closed-loop operator is $\check{A}_Q = -\sqrt{2+\kappa^2+2\kappa\cos\theta}$ with $\omega_0 = -\sqrt{2+\kappa^2-2\kappa}$. So

the eigenvalues of the closed-loop operator $\overline{A}_{Q_N}$ all lie in the spectrum of A_Q and the growth bound converges to ω_0 as $N \to \infty$. Moreover, $\|\overline{Q}_N\| \to \|\check{Q}\|$ as $N \to \infty$.
It is interesting to compare the above with the circular approximations. Following the approach in Example 3.11 we find the solution to (14) to be $\tilde{Q}_N = U_N \mathrm{diag}\left(1+\sqrt{1+\mu_k(N)}\right) U_N^*$, where the unitary matrix U_N is as in Example 3.11 and $\mu_k(N) = 1+\kappa^2+2\kappa\cos\frac{2k\pi}{2N+1}, k = 0,\ldots,2N$. Hence $\|\tilde{Q}_N\| = \max_{k=0,..,2N}(1+\sqrt{1+\mu_k(N)}) = 1+\sqrt{2+\kappa^2+2\kappa} = \|Q\|$. The closed-loop operator $\tilde{A}_{Q_N} = \tilde{A}_N - \tilde{B}_N\tilde{B}_N^*\tilde{Q}_N$ is given by

$$\tilde{A}_{Q_N} = U_N \mathrm{diag}\left(-\sqrt{2+\kappa^2+2\kappa\cos\frac{\pi}{2N+1}}\right) U_N^*$$

and its growth bound is $-\sqrt{2+\kappa^2-2\kappa\cos\frac{\pi}{2N+1}}$. So the eigenvalues of the closed-loop circulant approximating system all lie in the spectrum of A_Q and the growth bounds of their semigroups converge to ω_0 as $N \to \infty$.

5 Matlab simulations

Consider the following finite string of the form (1)

$$\begin{aligned}
\dot{z}_r(t) &= a_{-1}z_{r+1}(t) + a_0 z_r(t) + a_1 z_{r-1}(t) \\
&\quad + b_{-1}u_{r+1}(t) + b_0 u_r(t) + b_1 u_{r-1}(t), \ -N+1 \le r \le N, \\
\dot{z}_{-N}(t) &= a_{-1}z_{-N+1}(t) + a_0 z_{-N}(t) + b_{-1}u_{-N+1}(t) + b_0 u_{-N}(t), \\
\dot{z}_N(t) &= a_0 z_N(t) + a_1 z_{N-1}(t) + b_0 u_N(t) + b_1 u_{N-1}(t), \\
y_r(t) &= z_r(t), \quad -N \le r \le N, \quad t \ge 0.
\end{aligned}$$

which can be written as in (2) with $\mathbf{C}_N = I_N$,

$$\mathbf{A}_N = \begin{bmatrix} a_0 & a_{-1} & 0 & \ldots & 0 \\ a_1 & a_0 & a_{-1} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & a_1 & a_0 \end{bmatrix}, \mathbf{B}_N = \begin{bmatrix} b_0 & b_{-1} & 0 & \ldots & 0 \\ b_1 & b_0 & b_{-1} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & \ldots & b_1 & b_0 \end{bmatrix}.$$

Gaps between ω_N and ω_∞ are found in Example 5.1 case 4 and Example 5.2 case 4 ($T(\check{B})$ is not invertible). For $T(\check{B})$ invertible, we obtained in our simulations convergence not only for $\check{A} = \check{A}^*$ (Example 5.1 cases 1-3) but also also when $\check{A} \neq \check{A}^*$ (Example 5.2, except for case 4 in which $T(\check{B})$ is not invertible). We remark that when A is not self-adjoint the convergence rate of the growth bound is often slow (see Example 5.2, cases 1,3 and 8).

Example 5.1 *Consider* $\check{A}(e^{j\theta}) = \check{A}^*(e^{j\theta}) = a_0 + 2a_1\cos\theta$, $a_{-1} = a_1$.

case 1. $a_0 = 1,\ a_1 = a_{-1} = 0.3,\ b_0 = 1,\ b_1 = 0.1,\ b_{-1} = 0.2;$
case 2. $a_0 = 1,\ a_1 = a_{-1} = 0.3,\ b_0 = 1;$
case 3. $a_0 = 1,\ a_1 = a_{-1} = 0.3,\ b_0 = 1,\ b_1 = 0.1$
case 4. $a_0 = 1,\ a_1 = a_{-1} = 0.3,\ b_0 = 1,\ b_1 = 1.5$

Table 1
The growth bounds ω_∞ and ω_N for Example 5.1 ($\check{A} = \check{A}^*$)

case	ω_∞	ω_{10}	ω_{20}	ω_{40}	ω_{80}	ω_{160}
1.	-0.806	-0.812	-0.807	-0.806	-0.806	-0.806
2.	-1.077	-1.079	-1.077	-1.077	-1.077	-1.077
3.	-0.988	-0.985	-0.985	-0.985	-0.985	-0.984
4.	-0.640	-0.582	-0.582	-0.582	-0.582	-0.582

Table 2
Relative error ε_N (%) for Example 5.1 ($\check{A} = \check{A}^*$)

case	ε_{10}	ε_{20}	ε_{40}	ε_{80}	ε_{160}
1.	0.71	0.19	0.05	0.01	0.0034
2.	0.21	0.05	0.01	0.0039	0.0000
3.	0.35	0.09	0.02	0.006	0.0017
4.	-9.07	-9.07	-9.07	-9.07	-9.07

Example 5.2 $\check{A} \neq \check{A}^*$
case 1. $a_0 = -1,\ a_1 = 2,\ b_0 = 1.$
case 2. $a_0 = -1,\ a_1 = 2,\ b_0 = 1,\ b_1 = 0.4.$
case 3. $a_0 = 1,\ a_1 = 2,\ b_0 = 1,\ b_1 = 0.6.$
case 4. $a_0 = 1,\ a_1 = 2,\ b_0 = 1,\ b_1 = 1.6.$
case 5. $a_0 = 1,\ a_1 = 2,\ a_{-1} = 1,\ b_0 = 1.$
case 6. $a_0 = 1,\ a_1 = 2,\ a_{-1} = 1,\ b_0 = 1,\ b_1 = 0.4.$
case 7. $a_0 = 1,\ a_1 = 0.3,\ a_{-1} = 0.7,\ b_0 = 1.$
case 8. $a_0 = 1,\ a_1 = 0.3,\ a_{-1} = 0.7,\ b_0 = 1,\ b_1 = 0.4.$

Table 3
The growth bounds ω_∞ and ω_N for Example 5.2 ($\check{A} \neq \check{A}^*$)

case	ω_∞	ω_{10}	ω_{20}	ω_{40}	ω_{80}	ω_{160}
1.	-1	-1.151	-1.123	-1.114	-1.112	-1.080
2.	-1.232	-1.339	-1.317	-1.310	-1.309	-1.295
3.	-0.818	-1.060	-1.020	-1.009	-0.998	-0.930
4.	-1.148	-0.250	-0.250	-0.250	-0.250	-0.250
5.	-1	-1.072	-1.027	-1.012	-1.008	-1.007
6.	-0.959	-1.038	-0.990	-0.974	-0.970	-0.969
7.	-1	-1.111	-1.106	-1.104	-1.073	-1.038
8.	-0.6	-0.744	-0.741	-0.717	-0.669	-0.619

6 Conclusions

We have compared the growth bounds and the transient behavior of the LQR closed-loop operators of scalar finite strings with their infinite versions. Simple examples showed that stabilizability and detectability are not sufficient to ensure similar stability behavior of the LQR closed-loop strings as $N \to \infty$. For the circulant approximating systems this does hold. Under the stronger conditions of uniform stabilizability and detectability of the finite strings we can show that

Table 4
Relative error ε_N (%) for Example 5.2 ($\check{A} \neq \check{A}^*$)

case	ε_{10}	ε_{20}	ε_{40}	ε_{80}	ε_{160}	ε_{320}
1.	15.13	12.34	11.48	11.25	8.05	4.60
2.	8.64	6.86	6.32	6.17	5.05	2.94
3.	29.50	24.77	23.35	21.95	13.64	7.40
4.	-78.24	-78.24	-78.24	-78.24	-78.24	-78.24
5.	7.26	2.73	1.24	0.81	0.70	0.67
6.	8.23	3.24	1.63	1.18	1.06	1.04
7.	11.12	10.62	10.47	7.32	3.80	1.94
8.	24.04	23.50	19.63	11.52	6.36	3.17

the eigenvalues of the closed-loop approximating systems have an average distribution that is asymptotic to that of the infinite-dimensional system. We also give sufficient conditions under which the growth bound ω_N of different types of long-but-finite strings (such as circular configurations and systems for which two out of the three defining operators are constants) converges to the growth bound ω_∞ of the corresponding infinite string. However, in general is not true that ω_N converges to ω_∞. Similar results are obtained for an alternative sequence of Toeplitz approximating systems. Of course it is the MIMO case that is most interesting for applications, and this remains a challenging open problem. However, the scalar case has already demonstrated that LQR control of the infinite-dimensional strings does not always serve as a useful paradigm for the long-but-finite strings.

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A Notations and background on Toeplitz and Circulant matrices

Denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{C}$ and $\partial\mathbb{D}$ the sets of natural, integer, complex numbers and the unit circle, respectively. Let $\check{F} \in \mathbf{C} \subset \mathbf{L}_\infty(\partial\mathbb{D},\mathbb{C})$ be a continuous scalar symbol with the Fourier series representation $\check{F}(e^{j\theta}) = \sum_{l\in\mathbb{Z}} f_l e^{-jl\theta}$, $\theta \in [0,2\pi]$, where $\mathbf{C} := \mathbf{C}(\partial\mathbb{D},\mathbb{C})$ is the Banach algebra of all continuous functions on $\partial\mathbb{D}$ with the maximum norm ($\mathbf{L}_\infty$, $\mathbf{H}_\infty$, $\mathbf{L}_2$ and $\mathbf{H}_2$ are defined in [4]). Now, $\ell_2(\mathbb{Z},\mathbb{C}) = \{x \,|\, x = (x_r)_{r\in\mathbb{Z}}, x_r \in \mathbb{C}, \sum_{r\in\mathbb{Z}} |x_r|^2 < \infty\}$ is isometrically isomorphic to $\mathbf{L}_2(\partial\mathbb{D};\mathbb{C})$ ($\|x\|_{\ell_2(\mathbb{Z},\mathbb{C})} = \|\check{x}\|_{\mathbf{L}_2(\partial\mathbb{D},\mathbb{C})}$, $\check{x} = \mathfrak{F}x$) under the Fourier transform $\mathfrak{F}$. $\mathfrak{F}F\mathfrak{F}^{-1} = \check{F} : \mathbf{L}_2(\partial\mathbb{D},\mathbb{C}) \to \mathbf{L}_2(\partial\mathbb{D},\mathbb{C})$ is a multiplication operator generated the Laurent operator $F : l_2(\mathbb{Z},\mathbb{C}) \to l_2(\mathbb{Z},\mathbb{C})$, where F is defined by the convolution $((Fx)(t))_r = \sum_{\in\mathbb{Z}} a_l x_{r-l}(t) = \sum_{\in\mathbb{Z}} a_{r-l} x_l(t)$. Hence $\|F\| = \|\check{F}\|_\infty$ (we refer to [6], and the references therein, for further details).

Consider also the Toeplitz operator $T(\check{F}) : l_2(\mathbb{N},\mathbb{C}) \to l_2(\mathbb{N},\mathbb{C})$ given by

$$T(\check{F}) = \begin{bmatrix} f_0 & f_1 & f_2 & f_3 & \cdots \\ f_{-1} & f_0 & f_1 & f_2 & \cdots \\ f_{-2} & f_{-1} & f_0 & f_1 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}, \tag{A.1}$$

where $\ell_2(\mathbb{N},\mathbb{C}) = \{x \,|\, x = (x_r)_{r\in\mathbb{N}}, x_r \in \mathbb{C}, \sum_{r\in\mathbb{N}} |x_r|^2 < \infty\}$. We have $\|F\| = \|T(\check{F})\| = \|\check{F}\|_\infty$ and $\sigma(\check{F}) = \{\check{F}(e^{j\theta}), 0 \leq \theta \leq$

$2\pi\}$, $\sigma(T(\check{F})) = \sigma(\check{F}) \cup \{\lambda \notin \sigma(\check{F}) | \operatorname{wind}(\check{F}-\lambda, 0) \neq 0\}$, where $\operatorname{wind}(\check{F}-\lambda,0)$ is the winding number of $\check{F}-\lambda$ around the origin. Remark that F also has a matrix representation like (A.1) infinite in both directions.

We denote the *Toeplitz approximant* matrix of order n corresponding to F by $\mathbf{F}_n$

$$\mathbf{F}_n = \begin{bmatrix} f_0 & f_1 & f_2 & \cdots & f_{n-1} \\ f_{-1} & f_0 & f_1 & \cdots & f_{n-2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ f_{-n+1} & f_{-n+2} & f_{-n+3} & \cdots & f_0 \end{bmatrix} \tag{A.2}$$

The spectrum of $\mathbf{F}_n$ can be very different from that of $T(\check{F})$, except in the self-adjoint case.

Lemma A.1 Gray [8, Corollary 4.2]. *Suppose that $\check{F}$ is real. Denote by $m_{\check{F}}$ and $M_{\check{F}}$ the minimum and the maximum of $\check{F}$ on $[0,2\pi]$, respectively. If the Toeplitz approximants $\boldsymbol{F}_n$ have the eigenvalues $\lambda_k^{(n)}, k=1,...,n$. Then $m_{\check{F}} \le \lambda_k^{(n)} \le M_{\check{F}}$, $\lim_{n\to\infty} \max_k \lambda_k^{(n)} = M_{\check{F}}$, and $\lim_{n\to\infty} \min_k \lambda_k^{(n)} = m_{\check{F}}$.*

Proof: The proof in [8, Corollary 4.2] considers extra assumptions (summability), but this is only needed to establish (P8) below, which holds for all continuous symbols. ∎

We recall some properties of Toeplitz operators from in Böttcher and Silbermann [3], Theorems 1.15, 2.11, 4.3, 4.13.

Lemma A.2 *The singular values $\mu(\boldsymbol{F}_n)$ of $\boldsymbol{F}_n$ satisfy*

(1) $\lim_{n\to\infty} \mu_{\max}(\boldsymbol{F}_n) = \max_{\theta\in[0,2\pi]} |\check{F}(e^{j\theta})|$.

(2) $\lim_{n\to\infty} \mu_{\min}(\boldsymbol{F}_n) = 0$ *if and only if* $T(\check{F})$ *is not invertible.*

(3) $T(\check{F})$ *is invertible if and only if* $\check{F}(e^{j\theta}) \neq 0$, $\theta \in [0,2\pi]$, *and the winding number of f around the origin is zero.*

(4) *If* $\mu(\check{F}) \cap j\mathbb{R} = \emptyset$, *then* $T(\check{F})$ *is invertible and* $\lim_{n\to\infty} \mu_{\min}(\boldsymbol{F}_n) \neq 0$.

(5) *If* $T(\check{F})$ *is invertible, then* $\lim_{n\to\infty} \mu_{\min}(\boldsymbol{F}_n) = 1/\|T^{-1}(\check{F})\|$.

(6) *If* $T(\check{F})$ *is invertible, then* $\boldsymbol{F}_n$ *is invertible for* $n > n_0$ *and* $\sup_{n>n_0} \|\boldsymbol{F}_n^{-1}\| < \infty$.

Consider also $\tilde{F}_n$, the *circulant approximant matrix of order* n corresponding to $\check{F}$, given by

$$\tilde{F}_n = \begin{bmatrix} c_0^{(n)} & c_1^{(n)} & c_2^{(n)} & c_3^{(n)} & \cdots & c_{n-1}^{(n)} \\ c_{n-1}^{(n)} & c_0^{(n)} & c_1^{(n)} & c_2^{(n)} & \cdots & c_{n-2}^{(n)} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ c_1^{(n)} & c_2^{(n)} & c_3^{(n)} & c_4^{(n)} & \cdots & c_0^{(n)} \end{bmatrix}, \tag{A.3}$$

where $c_k^{(n)} = \frac{1}{n}\sum_{l=0}^{n-1} \check{F}\left(e^{j\frac{2\pi l}{n}}\right) e^{\frac{-2j\pi kl}{n}}$. Circulant approximant matrices have nice properties (see [8, Sections 3.1 and 3.2])

- (P1) $\|\tilde{F}_n\| \le \|F\|$.
- (P2) $\widetilde{(FG)}_n = \tilde{F}_n \tilde{G}_n$.
- (P3) $\widetilde{(F+G)}_n = \tilde{F}_n + \tilde{G}_n$.
- (P4) The eigenvalues of $\tilde{F}_n$ are $\lambda_k^{(n)} = \check{F}(e^{j\frac{2\pi k}{n}}), k = 0,1,...,n-1$.
- (P5) $\tilde{F}_n = U_n \operatorname{diag}(\lambda_k^{(n)}) U_n^*$, where U_n is unitary and has components $U_n(r,s) = \frac{1}{\sqrt{n}}\left[e^{-\frac{2\pi jrs}{n}}\right]$, $r,s = 0,...,n-1$.

In addition to the matrix spectral or induced $\mathbf{L}_2$-norm denoted by $\|\cdot\|$, following [8], we introduce the n-norm $|M|_n = \left(\frac{1}{n}\sum_{k=0}^{n-1}\sum_{l=0}^{n-1} |m_{kl}|^2\right)^{1/2} = \left(\frac{1}{n}\operatorname{trace}(M^*M)\right)^{1/2}$ for square matrices M of order n. This norm has the following properties (see[8, Lemma 2.3] and the references therein)

- (P6) $\frac{1}{\sqrt{n}}\|A\| \le |A|_n \le \|A\|$.
- (P7) $|AB|_n \le |A|_n \|B\| \quad |AB|_n \le \|A\| |B|_n$.
- (P8) $|\mathbf{F}_n - \tilde{F}_n|_n \to 0$ as $n \to \infty$ (see [9, Lemma 5])).

Note that Gutierrez quotes (P8) for self-adjoint matrices, but he does not use the self-adjoint property in his proof.

Lemma A.3 *Let $\check{F}$ be a scalar function with real values and $\{A_n\}$ be a sequence of self-adjoint $n \times n$ matrices such that $|A_n - \tilde{F}_n|_n \to 0$ as $n \to \infty$. Then*

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=0}^{n-1} \lambda(A_n)^2 = \frac{1}{2\pi}\int_0^{2\pi} \check{F}^2(e^{j\theta})\, d\theta.$$

Proof: Now $\tilde{F}_n^* \tilde{F}_n$ is also a circulant matrix with eigenvalues $|\check{F}(e^{2j\pi k/n})|^2$ and $|\tilde{F}_n|_n^2 = \frac{1}{n}\sum_{k=0}^{n-1} |\check{F}(e^{2j\pi k/n})|^2 \to \frac{1}{2\pi}\int_0^{2\pi} \check{F}^2(e^{j\theta})\, d\theta$ as $n \to \infty$, since $\check{F}$ is continuous and hence Riemann integrable. Hence $|A_n|_n^2 \to \frac{1}{2\pi}\int_0^{2\pi} \check{F}^2(e^{j\theta})\, d\theta$, as $n \to \infty$. The claims now follow from the classic asymptotic distribution theorem ([3]). ∎