New spectral graph theoretic conditions for synchronization in directed complex networks

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Abstract— **This paper proposes lower bounds for the coupling strengths of oscillators in directed networks to guarantee global synchronization. The novel idea of graph comparison from spectral graph theory is employed so that the topological features of a given network can be fully utilized to simplify computations. For large networks that can be decomposed into a set of smaller strongly connected components, the comparison can be carried out at the local level as well.**

I. Introduction

How to achieve synchronization in various complex networks has been a central research topic for decades. A systematic framework for the study of synchronization of nonlinear dynamical systems with diffusive couplings has been developed in [1]. One critical observation is that graph combinatorial features that are associated with the network topologies are essential for identifying synchronization conditions [2]. In particular, the lengths of all the paths passing through chosen edges in the graph have been used to allocate coupling strengths to achieve global synchronization in the network [3], [4].

We have designed in [5] a new coupling-strength allocation method for undirected networks using recently reported tools in spectral graph theory [6]. The idea of graph comparison [7], [6] turns out to be especially useful. In this paper, we further develop our methodologies by looking at general directed networks. We prove that the synchronization conditions given in [4] and [1] for allocating coupling strengths can be explained by comparing the network coupling graph with the corresponding complete graph. We construct an algorithm that incorporates graph comparison to find candidate sets of coupling strengths for synchronization. To keep the computation tractable, for large networks that can be decomposed into smaller strongly connected subgraphs, we run the algorithms locally for each component. The proposed algorithm is novel in that no graph comparison results for directed networks have been reported in the literature before and also that topological conditions are usually much easier to check.

The rest of the paper is organized as follows. In Section II, we formulate the system model. The graph comparison technique and the synchronization results obtained accordingly are presented in Section III. We deal with decomposable networks in Section IV.

II. System setup

We consider a network of $n > 1$ coupled identical oscillators whose dynamics are described by

$$\dot{x}_i = f(x_i) + \sum\nolimits_{j=1}^{n} \varepsilon_{ij}\, P\, x_j, \qquad i = 1, \ldots, n\,, \quad (1)$$

where $x_i \in \mathbb{R}^d$ is the state of the ith oscillator, $f(\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ denotes the identical self-dynamics of each oscillator and is a C^1 function, $\varepsilon_{ij} \geq 0$ describes the strength of the coupling from oscillator j to i, and the diagonal $(0,1)$-matrix $P \in \mathbb{R}^{d\times d}$ determines through which components of the states that the oscillators are coupled together. The couplings between the oscillators can be conveniently described by a graph $\mathbb{G}$ with n vertices in which there is an edge from vertex j to i if $\varepsilon_{ij} > 0$. Since the couplings between the oscillators are unidirectional, the network is directed in general and thus $\mathbb{G}$ is directed as well. The connectivity matrix C with entries ε_{ij} is an $n \times n$ matrix with zero row sums and nonnegative off-diagonal elements such that $\sum_{j=1}^{n} \varepsilon_{ij} = 0$ and $\varepsilon_{ii} = -\sum_{j=1, j\neq i}^{n} \varepsilon_{ij}$ for $i = 1, \ldots, n$.

System (1) has been used widely to study under what conditions the coupled oscillators can achieve asymptotically global and complete synchronization, where for any initial condition, $|x_i(t) - x_j(t)| \to 0$ as $t \to \infty$ for all i, j [1]. It is common to make one standard technical assumption about system (1).

Assumption 1: [4] For the network (1) consisting of two oscillators with one directed coupling, there exists some threshold a such that the global synchronization can be achieved if the coupling strength exceeds a.

Assumption 1 implies that any two coupled oscillators are always able to get synchronized when their coupling is sufficiently strong. Here the constant a is determined by both the function f and the projection matrix P.

In what follows, we look into how to apply spectral graph theory in order to gain new insight into the synchronization problem that we have just set up.

III. Synchronization criteria using spectral graph theory

There are existing results providing sufficient conditions for the global synchronization of system (1). We list two of such

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results below and they become useful later on as we develop spectral graph theoretic conditions in the paper. Let D_i^c denote the node unbalance [4] of node i, namely $D_i^c = \sum_{k=1}^{n} \varepsilon_{ki} = \sum_{k\neq i} \varepsilon_{ki} + \varepsilon_{ii} = \sum_{k\neq i} \varepsilon_{ki} - \sum_{k\neq i} \varepsilon_{ik}$, which is the difference between the out-degree and in-degree of node i.

Theorem 1: [4] Under Assumption 1, the synchronization manifold of system (1) is globally asymptotically stable if

$$\sum_{i=1}^{n-1} \sum_{j>i}^{n} \left(\frac{\varepsilon_{ij} + \varepsilon_{ji}}{2} \right) (x_{i_k} - x_{j_k})^2 > \frac{a}{n} \sum_{i=1}^{n-1} \sum_{j>i}^{n} \times \left(1 + \frac{1}{2a} (D_i^c + D_j^c) \right) (x_{i_k} - x_{j_k})^2 \tag{2}$$

for $1 \leq k \leq d$.

We use $\mathcal{W}_s$ to denote the set of irreducible, symmetric matrices that have zero row sums and non-positive off-diagonal elements. For a matrix $A \in \mathbb{R}^{n\times n}$, we say $A \succ 0$ if $x^T A x$ is positive for all nonzero $x \in \mathbb{R}^n$.

Theorem 2: (Chapter 4, [1]) Let $Y(t)$ be a $d \times d$ time-varying matrix and V be a $d \times d$ symmetric positive definite matrix such that $(y-z)^T V (f(y,t) + Y(t)y - f(z,t) - Y(t)z) \leq -c||y-z||^2$ for some $c > 0$ and all y, z, t. Then network (1) synchronizes globally if there exists an $n \times n$ matrix $U \in \mathcal{W}_s$ such that

$$(U \otimes V)(G(t) \otimes D(t) - I_n \otimes Y(t)) \preceq 0\,, \quad \text{for all } t. \tag{3}$$

Here, the time-varying matrices $D(t)$ and $G(t)$ correspond to $-P$ and $L_{\mathbb{G}}$ in our model (1). In our related work [5], we have shown that Assumption 1 is equivalent to the condition $(y-z)^T (f(y,t) + Y(t)y - f(z,t) - Y(t)z) \leq -c||y-z||^2$ when we set $Y(t) = -aP$.

In this paper, we intend to introduce tools from spectral graph theory to study the synchronization problem. To do so, we need to define some notations. For a symmetric square matrix A, by $A \succ 0$ we mean that A is positive definite. And we say $A \succ B$ if $A - B \succ 0$. Similarly, we say $A \succeq B$ if $A - B$ is positive semi-definite. We further apply this notation to undirected graphs.

Definition 1: For two undirected graphs $\mathbb{H}_1$ and $\mathbb{H}_2$ with the same vertex set $\mathcal{V} = \{1, \ldots, n\}$, we say $\mathbb{H}_1 \succeq \mathbb{H}_2$ if their Laplacian matrices satisfy $L_{\mathbb{H}_1} \succeq L_{\mathbb{H}_2}$.

We use $\mathbb{K}_n$ to denote the un-weighted, undirected complete graph with n vertices. To apply Theorem 2, we set $Y(t) = aD(t), V = I_d, U = L_{\mathbb{K}_n}, G(t) = L_{\mathbb{G}}$. Then from (3) we have $(L_{\mathbb{K}_n} \otimes I_d)(L_{\mathbb{G}} \otimes D(t) - I_n \otimes aD(t)) \preceq 0$, i.e., $L_{\mathbb{K}_n} L_{\mathbb{G}} \otimes D(t) - aL_{\mathbb{K}_n} \otimes D(t) \preceq 0$. Since $D(t) \preceq 0$, we have $L_{\mathbb{K}_n} L_{\mathbb{G}} - aL_{\mathbb{K}_n} \succeq 0$. Therefore the complete synchronization of network (1) is guaranteed if

$$L_{\mathbb{K}_n} L_{\mathbb{G}} \succ a\, L_{\mathbb{K}_n}\,. \tag{4}$$

Note that $L_{\mathbb{K}_n} = nI_n - J$ where J is the n-by-n all-one matrix. One has $n\mathbb{G} - J\mathbb{G} \succ a\mathbb{K}_n$ from (4). In the case where $\mathbb{G}$ is undirected, (4) can be further reduced to $\mathbb{G} \succ \frac{a}{n}\mathbb{K}_n$. However, in this paper we are looking at the more challenging scenario where the networks are directed. We will use the following property of directed graphs.

Lemma 1: For a directed graph $\mathbb{G}$, the condition $n\mathbb{G} - J\mathbb{G} \succ a\mathbb{K}_n$ is equivalent to $\frac{n}{2}(\mathbb{G} + \mathbb{G}^T) - \frac{1}{2}(J\mathbb{G} + \mathbb{G}^T J) \succ a\mathbb{K}_n$.

We omit the proof for this lemma due to page limit.

With Theorem 2 and Lemma 1 at hand, we have arrived at a general graph theoretic synchronization criterion.

Theorem 3: Suppose that Assumption 1 holds, and that the graph $\mathbb{G}$ contains a directed spanning tree [2]. The synchronization manifold of system (1) is globally asymptotically stable if $\frac{n}{2}(\mathbb{G} + \mathbb{G}^T) - \frac{1}{2}(J\mathbb{G} + \mathbb{G}^T J) \succ a\mathbb{K}_n$.

In the following, we will show that Theorem 1 is equivalent to Theorem 3. It holds that

$$\begin{aligned} J\,\mathbb{G} &= \mathbf{1} \otimes \begin{bmatrix} -\sum_{k=1}^{n} \varepsilon_{k1} & -\sum_{k=1}^{n} \varepsilon_{k2} & \ldots & -\sum_{k=1}^{n} \varepsilon_{kn} \end{bmatrix} \\ &= -\mathbf{1} \otimes \begin{bmatrix} D_1^c & D_2^c & \ldots & D_n^c \end{bmatrix}, \end{aligned}$$

where $\mathbf{1} \in \mathbb{R}^n$ is the vector of all ones. And one has

$$\mathbb{G}^T J = -\mathbf{1}^T \otimes \begin{bmatrix} D_1^c & D_2^c & \ldots & D_n^c \end{bmatrix}^T.$$

It follows that the matrix $-(J\,\mathbb{G} + \mathbb{G}^T J)$ is

$$\begin{bmatrix} 2D_1^c & D_1^c + D_2^c & \ldots & D_1^c + D_n^c \\ D_2^c + D_1^c & 2D_2^c & \ldots & D_2^c + D_n^c \\ \ldots & \ldots & \ldots & \ldots \\ D_n^c + D_1^c & D_n^c + D_2^c & \ldots & 2D_n^c \end{bmatrix},$$

where the (ij)th entry is $D_i^c + D_j^c$ for $i, j = 1, \ldots, n$. Since the sum of the out-degrees of all the nodes in $\mathbb{G}$ is equal to the sum of the in-degrees of all the nodes, we have $\sum_{i=1}^{n} D_i^c = 0$. The ith row-sum of the matrix $-(J\,\mathbb{G} + \mathbb{G}^T J)$ is then $nD_i^c + \sum_{i=1}^{n} D_i^c = nD_i^c$ for $i = 1, \ldots, n$. Let the $n \times n$ matrix $\Delta \triangleq \text{diag}\{nD_1^c, nD_2^c, \ldots, nD_n^c\}$. Thus, the matrix $a\mathbb{K}_n + \frac{1}{2}(J\mathbb{G} + \mathbb{G}^T J) + \frac{1}{2}\Delta$ is symmetric and has zero row sums. Since the ith row-sum of the matrix $\mathbb{G} + \mathbb{G}^T$ is $-\sum_{k=1}^{n} \varepsilon_{ki} = -D_i^c$ for $i = 1, \ldots, n$, we know that the matrix $\frac{n}{2}(\mathbb{G} + \mathbb{G}^T) + \frac{1}{2}\Delta$ is symmetric and has zero row sums and non-positive off-diagonal entries. Now we are ready to compare the two symmetric matrices $a\mathbb{K}_n + \frac{1}{2}(J\mathbb{G} + \mathbb{G}^T J) + \frac{1}{2}\Delta$ and $\frac{n}{2}(\mathbb{G} + \mathbb{G}^T) + \frac{1}{2}\Delta$. From Theorem 3, we have

$$\frac{1}{2}(\mathbb{G} + \mathbb{G}^T) + \frac{1}{2n}\Delta \succ \frac{a}{n}\left(\mathbb{K}_n + \frac{1}{2a}(J\mathbb{G} + \mathbb{G}^T J) + \frac{1}{2a}\Delta \right). \tag{5}$$

Furthermore, one can easily check that the inequality (5) is equivalent to the inequality (2) in Theorem 1. Therefore, we have shown that Theorem 3 and Theorem 1 are one and the same.

To apply more tools from spectral graph theory, we need to introduce another equivalent definition of Laplacian matrices of graphs. Following [7], we define the elementary Laplacian $L_{(u,v)}$ to be the Laplacian of the graph with the vertex set $\mathcal{V}$ and only one edge between vertices u and v. Then for an arbitrary undirected graph $\mathbb{H}$ with the edge set $\mathcal{E}$, its Laplacian matrix can be defined to be $L(\mathbb{H}) \triangleq \sum_{(u,v)\in\mathcal{E}} L_{(u,v)}$.

The following inequality have been proved in [7] and we list it below.

Lemma 2: [7] For the undirected graph $\mathbb{H}$ with edge set $\mathcal{E}$, it holds that $(n-1)\left(\sum_{i=1}^{n-1} L_{(i,i+1)}\right) \succeq L_{(1,n)}$.

The spectral graph theory discussed in [7] mainly focuses on undirected graphs. It has been demonstrated in [5] that tools in spectral graph theory are powerful in utilizing flexibly topological features of a given network. However, the results developed in [5] can be applied only to undirected networks and are thus not general enough. Motivated by [4], what we propose to do in the next is to symmetrize graph $\mathbb{G}$ first, and then construct synchronization criteria on the symmetrized graph using spectral graph theory. To be more specific, for any pair of unidirectionally coupled nodes i and j, we replace the directed edge between them by an undirected edge with the weight $\varepsilon_{ij}/2$ that is half of the original coupling strength; for any bi-directionally coupled pair of nodes i and j, we replace the two edges between them by an undirected edge with the coupling strength $(\varepsilon_{ij}+\varepsilon_{ji})/2$. Let $\mathbb{G}^s$ denote the obtained symmetrized graph from $\mathbb{G}$. One can then check that the Laplacian matrix of $\mathbb{G}^s$ is $L_{\mathbb{G}^s}=\frac{1}{2}\left(L_{\mathbb{G}}+L_{\mathbb{G}}^T\right)+\frac{1}{2n}\Delta$.

For the symmetrized graph $\mathbb{G}^s$, consider a set of paths $\mathcal{P}=\{\mathcal{P}_{ij}|i,j=1,\ldots,n,j>i\}$, one for each pair of distinct nodes i and j. We denote the length of the path $\mathcal{P}_{ij}$ by $|\mathcal{P}_{ij}|$, which is the number of edges in $\mathcal{P}_{ij}$.

Now we use Theorem 3 and (5) to construct graph theoretic conditions for the synchronization of network (1). We use $\mathcal{E}(\mathbb{G}^s)$ to denote the set of all the edges of $\mathbb{G}^s$ and assume that there are altogether m edges that are labeled by $1,\ldots,m$. In the following theorem, we show that lower bounds on the coupling strengths $\varepsilon_k, k=1,\ldots,m$, can be constructed to guarantee that the inequality (5) holds.

Theorem 4: Suppose that Assumption 1 holds, and the graph $\mathbb{G}$ contains a directed spanning tree. The synchronization manifold of network (1) is globally asymptotically stable if

$$\varepsilon_k > \frac{a}{n}\, b_k, \qquad \text{for } k=1,\ldots,m, \tag{6}$$

where $b_k=\sum_{j>i;k\in\mathcal{P}_{ij}} L(\mathcal{P}_{ij})$ is the sum of the lengths $L(\mathcal{P}_{ij})$ of all those paths $\mathcal{P}_{ij}$ in $\mathcal{P}$ containing the edge k that belongs to the symmetrized graph $\mathbb{G}^s$ and the weighted path length $L(\mathcal{P}_{ij})$ is defined by

$$L(\mathcal{P}_{ij}) \triangleq \begin{cases} |\mathcal{P}_{ij}|\,\chi\left(1+\frac{D_i^c+D_j^c}{2a}\right), & \text{edge}(i,j)\notin\mathcal{E}(\mathbb{G}^s)\,; \\ 1+\frac{D_i^c+D_j^c}{2a}\,, & \text{edge}(i,j)\in\mathcal{E}(\mathbb{G}^s)\,, \end{cases} \tag{7}$$

where the function $\chi(\cdot)$ returns identity for positive arguments and 0 otherwise.

Theorem 4 can be obtained through comparison of the two matrices $L_{\mathbb{G}^s}$ and $\frac{a}{n}\left(L_{\mathbb{K}_n}+\frac{1}{2a}\left(JL_{\mathbb{G}}+L_{\mathbb{G}}^T J\right)+\frac{1}{2a}\Delta\right)$, using Lemma 2. We omit the proof for Theorem 4 due to page limit.

Remark 1: Using graph comparison, we have provided a different proof compared with those in [8], [4]. Our approach utilizes more the features of the graphs associated with the networks.

Remark 2: If $\mathbb{G}$ is asymmetric but balanced, then $D_i^c=0$ for $i=1,\ldots,n$. From Theorem 4, it follows that network (1) can be asymptotically synchronized if $\varepsilon_k>\frac{b_k}{n}\,a$ for $k=1,\ldots,m$, where $b_k=\sum_{j>i;k\in\mathcal{P}_{ij}}|\mathcal{P}_{ij}|$. The result then becomes the same as Theorem 1 in [8], in which the connection graph stability method is discussed for directed graphs with node balance.

Theorem 4 can be used to find a set of coupling strengths to realize global synchronization in a network. We describe below an algorithm to achieve this goal.

Step 1. Determine the node unbalance D_i^c for each node.
Step 2. Symmetrize $\mathbb{G}$ to obtain the undirected graph $\mathbb{G}^s$.
Step 3. Compare $\mathbb{G}^s$ with the corresponding complete graph $\mathbb{K}_n$. For any pair of nodes i,j, choose a path $\mathcal{P}_{ij}$ in $\mathbb{G}^s$. Here, we prefer to choose the shortest paths.
Step 4. For those paths $\mathcal{P}_{ij}$ whose lengths are greater than 1, assign the weight $1+\frac{D_i^c+D_j^c}{2a}$ if $1+\frac{D_i^c+D_j^c}{2a}>0$, and zero otherwise. For those paths $\mathcal{P}_{ij}$ whose lengths equal 1, assign the weight $1+\frac{D_i^c+D_j^c}{2a}$.
Step 5. For each edge k in $\mathbb{G}^s$ write down the inequality (6).
Step 6. Solve for the solutions to the obtained set of inequalities, which give possible combinations of coupling strengths.

Remark 3: Similar ideas in this algorithm have been discussed in [4]. The main differences lie in Steps 4 and 5 where we have used graph comparison techniques. Following [4], we call this algorithm the generalized connection graph method and use the abbreviation GCGM in the rest of the paper.

In the next section, we discuss in more detail a new systematic way to allocate coupling strengths for large networks with local structures.

IV. Networks with local structures

Although GCGM uses the combinatorial features of graphs and sometimes can greatly simplifies computation, it still has two shortcomings:

1) The computational complexity of counting paths grows exponentially as the size of the network inceases.
2) As the number of inequalities obtained in step 5 increases, it becomes more difficult, sometimes impossible, to find a solution in step 6.

To address these two shortcomings, we improve our results by looking more carefully at the networks' local structures and thus apply graph comparison only locally. To do so, we need to decompose graphs.

Definition 2: [1] The Frobenius normal form of the Laplacian matrix of a directed graph $\mathbb{G}$ is:

$$L_{\mathbb{G}} = M \begin{bmatrix} B_1 & B_{12} & \ldots & B_{1k} \\ & B_2 & \ldots & B_{2k} \\ & & \ddots & \vdots \\ & & & B_k \end{bmatrix} M^T \tag{8}$$

where M is a permutation matrix and B_i are square irreducible matrices.

Lemma 3: [1] The matrices B_i are uniquely determined by $L_{\mathbb{G}}$ although their ordering can be arbitrary as long as they follow a partial order induced by $\lhd$ which is defined as $B_i \lhd B_j \Leftrightarrow B_{ij}\neq 0$.

The uniqueness of the matrices B_i can be seen from the fact that these matrices correspond to the strongly connected

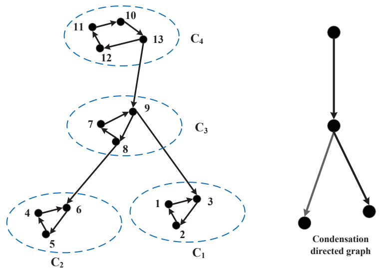

Fig. 1. A directed graph and its condensation directed graph.

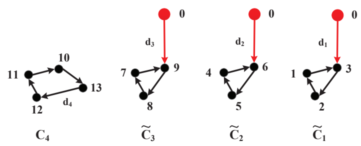

Fig. 2. Graph components.

components of graph $\mathbb{G}$. The decomposition of a Laplacian matrix into its Frobenius normal form is thus equivalent to the decomposition of $\mathbb{G}$ into its strongly connected components. The partial order in Lemma 3 leads to the definition of condensation directed graphs as follows.

Definition 3: [1] The condensation directed graph of a directed graph $\mathbb{G}$ is constructed by assigning a vertex to each strongly connected component of $\mathbb{G}$ and an edge between two vertices if and only if there exists an edge of the same orientation between corresponding strongly connected components of $\mathbb{G}$.

We give an example of a directed graph and its corresponding condensation graph in Fig. 1. We will also need the following definition for the generalized Fielder's algebraic connectivity and its property.

Definition 4: [1] For a directed graph with the associated Laplacian matrix L expressed in its Frobenius normal form, define $a_4(L) = \min_{1 \le i \le k} \eta_i$ where $\eta_i = \min_{x \neq 0} \frac{x^T W_i B_i x}{x^T W_i x}$ for $1 \le i \le k-1$ and $\eta_k = \min_{x \neq 0, x \perp \mathbf{1}} \frac{x^T W_k B_k x}{x^T \left(W_k - \frac{w_k w_k^T}{\|w_k\|_1} \right) x}$.

The following synchronization criteria can be derived directly from Theorem 4.20 and Corollary 4.21 in [1].

Theorem 5: Under the assumptions of Theorem 2, if $\mathbb{G}$ contains a directed spanning tree, then the network (1) synchronizes for sufficiently large coupling strength $\varepsilon\, a_4(L_{\mathbb{G}}) > a$.

Theorem 5 enables us to design the following algorithm to obtain the sets of coupling strengths for global synchronization using only local topological structure information. To avoid notational confusion and distinguish from the coupling strengths obtained by GCGM, we use d to denote the coupling strength to be found using graph decomposition.

Step 1. Decompose $\mathbb{G}$ into its k, $k \le n$, strongly connected components $\mathbb{C}_1, \mathbb{C}_2, \ldots, \mathbb{C}_k$ and the partial ordering is given by Lemma 3.

Step 2. For $\mathbb{C}_k$, use the GCGM algorithm in Section III to obtain a lower bound of the coupling strength d_k to synchronize the nodes in $\mathbb{C}_k$.

Step 3. In descending order for $i = k-1, \ldots, 1$, treat $\mathbb{C}_i$ one by one. Replace all those nodes in $\mathbb{C}_{i+1}, \ldots, \mathbb{C}_k$, by a single node 0. And keep the edges between $\mathbb{C}_i$ and $\mathbb{C}_{i+1}, \ldots, \mathbb{C}_k$. Thus we arrive at an condensed component $\tilde{\mathbb{C}}_i$. Use the GCGM algorithm to obtain a lower bound of the coupling strength d_i for synchronization in $\tilde{\mathbb{C}}_i$.

Step 4. Combine d_i to get d.

We use an example to show the effectiveness of the algorithm. We consider the directed network on the left of Fig. 1. For simplicity, we choose to use an identical coupling strength in the network. We follow all the four steps. First, we decompose $\mathbb{G}$ into $\mathbb{C}_1, \mathbb{C}_2, \mathbb{C}_3, \mathbb{C}_4$ as shown in Fig. 1. And thus we have the partial orderings $B_2 \lhd B_3 \lhd B_4$ and $B_1 \lhd B_3 \lhd B_4$. The condensation graph is shown on the right of Fig. 1. For $\mathbb{C}_4$, using the GCGM algorithm, we calculate that $d_4 > \frac{3}{2}\, a$. For $\mathbb{C}_3$, we obtain $\tilde{\mathbb{C}}_3$ shown in Fig. 2 and use the GCGM algorithm to obtain $d_3 > 3a$. Similarly, we get $d_2 > 3a$ for $\tilde{\mathbb{C}}_2$ and $d_1 > 3a$ for $\tilde{\mathbb{C}}_1$. Finally, taking the maximum over d_1 to d_4 together, we conclude that the global synchronization of the network $\mathbb{G}$ can be realized under the coupling strength $d > 3a$.

V. Conclusion

In this paper we have presented new ways to allocate coupling strengths using spectral graph theory in order to achieve synchronization in directed complex networks. The main idea is to use graph comparison. The obtained results can be applied to large but decomposable networks as well.

For future study, we are interested in using the constructed synchronization criteria to develop optimal or sub-optimal solutions to add or delete edges in a network to achieve better synchronizability.

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