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Clustering in diffusively coupled networks ☆

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ABSTRACT

This paper shows how different mechanisms may lead to clustering behavior in connected networks consisting of diffusively coupled agents. In contrast to the widely studied synchronization processes, in which the states of all the coupled agents converge to the same value asymptotically, in the cluster synchronization problem studied in this paper, we require all the interconnected agents to evolve into several clusters and each agent only to synchronize within its cluster. The first mechanism is that agents have different self-dynamics, and those agents having the same self-dynamics may evolve into the same cluster. When the agents' self-dynamics are identical, we present two other mechanisms under which cluster synchronization might be achieved. One is the presence of delays and the other is the existence of both positive and negative couplings between the agents. Some sufficient and/or necessary conditions are constructed to guarantee n-cluster synchronization. Simulation results are presented to illustrate the effectiveness of the theoretical analysis.

1. Introduction

Recently, the study of distributed coordination of multi-agent systems has attracted significant attention from researchers from diverse backgrounds. Simple local coordination rules can sometimes lead to complicated collective behavior, such as the synchronization that has been discovered in natural, social, and engineered networks and systems (Strogatz, 2003). Various algorithms have been successfully constructed to cause all the agents in a group to converge to the same value asymptotically (Cao, Morse, & Anderson, 2008; Jadbabaie, Lin, & Morse, 2003; Ren & Beard, 2005). At the same time, there is an emerging trend to study how an interconnected group may incorporate or evolve into different subgroups, called clusters. In nature, multi-species foraging groups have been observed, such as flocks of bark-foraging birds (Dolby & Grubb, 1998), in which birds have to coordinate through interactions with peers in their own and other species. In the study of social networks, some opinion dynamics models (Hegselmann & Krause, 2002) describe how the agents with bounded confidence levels evolve into different clusters, where the agents in the same cluster hold the same opinion in the end. The clustering behavior is also potentially useful for the formation control problem for teams of autonomous agents (Anderson, Yu, Fidan, & Hendrickx, 2008).

Motivated by the reported clustering phenomena, we aim to study in this paper the cluster synchronization problem, in which a coupled multi-agent system is required to split into several clusters, so that the agents synchronize with one another in the same cluster, but differences exist between different clusters (Xia & Cao, 2010). Here, the model we adopt is obtained by carrying out a modification to the existing synchronization model that has been used extensively to explain how the states of all the coupled agents converge to the same value asymptotically. In other words, we are interested in identifying the mechanisms that might lead to clustering behavior in diffusively coupled networks that have mainly been used for synchronization studies. Such problems are beginning to attract attention. For example, in Wu, Zhou, and Chen (2009), some sufficient conditions have been derived for coupled oscillators to realize cluster synchronization under pinning control strategies. In this paper, we focus on the n-cluster synchronization problem to be defined in the next section. We present three different mechanisms to realize clustering behavior in connected diffusively coupled networks. One is the existence of different self-dynamics and the other two are the presence of delays and negative couplings, respectively. When analyzing the three mechanisms, we also list related results that are scattered in the literature and make comparisons when possible.

The rest of the paper is organized as follows. We define n-cluster synchronization in Section 2. Different sufficient and/or necessary conditions to guarantee cluster synchronization under different

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mechanisms are discussed in Sections 3 and 4. In Section 5, we provide some illustrative examples.

2. Cluster synchronization

We first give a formal definition of n-cluster synchronization. Consider the following extensively studied model in the synchronization study of a complex network (Kocarev & Parlitz, 1996; Lu & Chen, 2004a,b; Pecora & Carroll, 1990) that consists of N coupled agents:

$$\begin{aligned}\dot{x}_i(t) &= f_i(t, x_i(t)) + c\sum_{j=1, j\neq i}^{N} g_{ij}\Gamma(x_j(t) - x_i(t)) \\ &= f_i(t, x_i(t)) + c\sum_{j=1}^{N} g_{ij}\Gamma x_j(t), \end{aligned} \tag{1}$$

where $x_i \in \mathbb{R}^m$ denotes the state of agent i, $i = 1, \ldots, N$, f_i: $\mathbb{R}^+ \times \mathbb{R}^m \to \mathbb{R}^m$ is continuous and globally Lipschitzian with Lipschitz constant K_i, namely

$$\|f_i(t, \xi_1) - f_i(t, \xi_2)\| \leq K_i\|\xi_1 - \xi_2\|, \tag{2}$$

for all (t, ξ_1), $(t, \xi_2) \in \mathbb{R}^+ \times \mathbb{R}^m$ with $\|\cdot\|$ denoting the Euclidean norm, $c > 0$ is the coupling strength, g_{ij} is the coefficient for the coupling from agent j to agent i for $i \neq j$, $i, j = 1, \ldots, N$, $g_{ii} = -\sum_{j=1, j\neq i}^{N} g_{ij}$, and the diagonal matrix $\Gamma = \mathrm{diag}\{\gamma_1, \ldots, \gamma_m\}$ denotes the inner coupling with $\gamma_k \geq 0$ for $k = 1, \ldots, m$. System (1) has a unique solution which exists for all $t \geq 0$ (Driver, 1977). The condition $g_{ii} = -\sum_{j=1, j\neq i}^{N} g_{ij}$ guarantees that the inter-agent couplings are diffusive, and hence such networks are also called *diffusively coupled networks*.

Directed weighted graphs can be conveniently used to describe the couplings between agents. For the matrix $G = (g_{ij})_{N\times N}$ whose elements are the same as defined in (1), we define its associated graph $\mathbb{G}$ to be the directed weighted graph with the vertex set $\mathcal{V}(\mathbb{G}) = \{v_1, v_2, \ldots, v_N\}$ and the edge set $\mathcal{E}(\mathbb{G}) \subset \{(v_i, v_j) : v_i, v_j \in \mathcal{V}(\mathbb{G})\}$ for which (v_i, v_j) is an edge of $\mathbb{G}$ if and only if $i \neq j$ and $g_{ji} \neq 0$, and the weight associated with (v_i, v_j) is g_{ji}. A directed path in $\mathbb{G}$ is a sequence of distinct vertices $v_{i_1}, \ldots, v_{i_k}$ such that $(v_{i_s}, v_{i_{s+1}}) \in \mathcal{E}(\mathbb{G})$ for $s = 1, \ldots, k-1$. $\mathbb{G}$ is said to be *strongly connected* if there is a directed path from every vertex to every other vertex in $\mathbb{G}$, and it is said to be *balanced* if $\sum_{j=1}^{N} g_{ij} = \sum_{j=1}^{N} g_{ji}$ for all i.

We say that $\{C_1, C_2, \ldots, C_n\}$, $n > 1$, is a *partition* of the set $\mathcal{N} = \{1, 2, \ldots, N\}$ if $C_i \neq \emptyset$, $C_i \bigcap C_j = \emptyset$, and $\bigcup_{i=1}^{n} C_i = \mathcal{N}$; furthermore, we use $\hat{i}$ to denote the index of that subset of the partition in which the number i lies, i.e., $i \in C_{\hat{i}}$. Obviously, $1 \leq \hat{i} \leq n$. We say that agents i and j are in the same cluster if $\hat{i} = \hat{j}$. Now, we are ready to define what we mean by cluster synchronization.

Definition 1. For a given initial condition $x(0) = [x_1^T(0), \ldots, x_N^T(0)]^T$, where $x_i(0) \in \mathbb{R}^m$, $1 \leq i \leq N$, system (1) is said to realize n-cluster synchronization with the partition $\{C_1, C_2, \ldots, C_n\}$ if $\lim_{t\to\infty} \|x_i(t) - x_j(t)\| = 0$ for $\hat{i} = \hat{j}$ and $\limsup_{t\to\infty} \|x_i(t) - x_j(t)\| > 0$ for $\hat{i} \neq \hat{j}$.

Remark 1. In Yu and Wang (2009), a similar concept called the "group consensus" of a multi-agent system is defined, which is weaker than the cluster synchronization defined here, because we require in addition that the differences between different clusters do not go to 0 as time goes to infinity. A different type of clustering behavior is considered in Aeyels and De Smet (2009) and De Smet and Aeyels (2009), where the differences between agents in the same cluster are bounded, while the differences between agents in different clusters grow unbounded as time goes to infinity.

In the synchronization study literature, the f_i in (1) are always referred to as the *self-dynamics* of agent i. In what follows, we discuss clustering mechanisms according to whether the agents' self-dynamics are identical.

3. Clustering with different self-dynamics

We first illustrate how agents governed by different linear dynamics might evolve into different clusters. We consider the case when some agents are under constant forcings and the others are not. The dynamics of the former are

$$\dot{x}_i(t) = -x_i(t) + a_{\hat{i}} + \sum_{j=1}^{N} g_{ij}x_j(t), \tag{3}$$

where $g_{ij} \geq 0$ for $i \neq j$, $\sum_{j=1}^{N} g_{ij} = 0$, and the $a_{\hat{i}}$ are constants with $a_{\hat{i}} \neq a_{\hat{j}}$ for $\hat{i} \neq \hat{j}$. The dynamics of the latter are

$$\dot{x}_i(t) = \sum_{j=1}^{N} g_{ij}x_j(t), \tag{4}$$

where the g_{ij} satisfy the same constraints as for (3). Comparing (3) and (4) with (1), we have taken the f_i to be linear functions, Γ an identity matrix, $c = 1$, and $m = 1$. The results derived in this section can be easily extended to the more general case when $c > 0$ and $m \geq 1$. Since the constant forcing terms sometimes come from the agents' knowledge about their preferred values, the agents described by (3) are called *informed agents*, and naturally the agents described by (4) are called *naive agents* since they do not have prior knowledge and have to rely on the interactions with their peers to evolve. In the next two subsections, we provide some sufficient and/or necessary conditions for systems of informed and naive agents to converge to n clusters.

3.1. Systems of informed agents

In this subsection, we consider the case when the system only consists of N informed agents described by (3) for $1 \leq i \leq N$. Assume that we have labeled the agents in such a way that the first l_1 agents are under the forcing a_1, the next l_2 agents are under a_2, and so on. Then the system can be written in a compact form:

$$\dot{x}(t) = -x(t) + \bar{a} + Gx(t) = \bar{G}x(t) + \bar{a}, \tag{5}$$

where $x = [x_1, x_2, \ldots, x_N]^T \in \mathbb{R}^N$, $G = (g_{ij})_{N\times N}$, $\bar{G} = G - I$, and $\bar{a} = [a_1\mathbf{1}_{l_1}^T, \ldots, a_{n-1}\mathbf{1}_{l_{n-1}}^T, a_n\mathbf{1}_{l_n}^T]^T$. Here I is the identity matrix, $\mathbf{1}_{l_k}$ are the l_k-dimensional all-one column vectors for $k = 1, \ldots, n$, and $l_1 + \cdots + l_n = N$.

We further write the matrix G in the following block matrix form:

$$G = \begin{bmatrix} G_{11} & G_{12} & \cdots & G_{1n} \\ G_{21} & G_{22} & \cdots & G_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n1} & G_{n2} & \cdots & G_{nn} \end{bmatrix},$$

where $G_{ij} \in \mathbb{R}^{l_i\times l_j}$, $1 \leq i, j \leq n$. Since the row sums of G are zero, we know the row sums of $\bar{G}$ are -1. In addition, $\bar{G}$ has non-negative off-diagonal elements. Hence, $\bar{G}$ is invertible, and the eigenvalues of $\bar{G}$ are all located in the open left-half plane. The equilibrium of system (5) is $x^* = -\bar{G}^{-1}\bar{a}$. Define $y(t) = x(t) - x^*$; then $\dot{y}(t) = \bar{G}y(t)$. It is obvious that $y(t) \to 0$ as $t \to \infty$. Thus x^* is a globally stable equilibrium of system (5). In fact, we can say more about the structures of x^* as follows.

Theorem 1. *For any initial condition, system (5) of informed agents achieves n-cluster synchronization for almost all (in the sense of Lebesgue measure) a_i, $1 \le i \le n$, with $a_i \ne a_j$ for $i \ne j$, if the block matrices G_{ij}, $1 \le i, j \le n$ and $i \ne j$, have constant row sums.*

The proof of this theorem makes use of the following lemma.

Lemma 1. *Consider the matrix $P = (P_{ij})_{N\times N}$, where $P_{ij} \in \mathbb{R}^{l_i \times l_j}$, $1 \le i, j \le n$. Suppose that P is invertible and that its inverse is $Q = (Q_{ij})_{N\times N}$, where Q is partitioned in the same way as P. If the matrices P_{ij} have constant row sums for $1 \le i, j \le n$, then the matrices Q_{ij} also have constant row sums for $1 \le i, j \le n$. In addition, let r_{ij} denote the row sum of P_{ij} and s_{ij} denote that of Q_{ij}; then $RS = I_{n\times n}$, where $R = (r_{ij})_{n\times n}$ and $S = (s_{ij})_{n\times n}$.*

Proof. From $QP = I$, one has

$$\sum_{k=1}^{n} Q_{1k}P_{kj} = \begin{cases} I, & j = 1, \\ O, & j \ne 1, \end{cases}$$

where O is the zero matrix of appropriate dimension. Since the P_{kj} have constant row sums r_{kj}, summing up the elements in each row of the P_{kj} gives

$$\left(\begin{bmatrix} r_{11} & r_{21} & \cdots & r_{n1} \\ r_{12} & r_{22} & \cdots & r_{n2} \\ \vdots & \vdots & \ddots & \vdots \\ r_{1n} & r_{2n} & \cdots & r_{nn} \end{bmatrix} \otimes I\right) \begin{bmatrix} Q_{11}\mathbf{1} \\ Q_{12}\mathbf{1} \\ \vdots \\ Q_{1n}\mathbf{1} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \tag{6}$$

where $\mathbf{1}$ and $\mathbf{0}$ are the all-one and all-zero column vectors of appropriate dimensions, respectively, and $\otimes$ denotes the Kronecker product. Since P is invertible, so is R. Combining with (6), we know that the Q_{1j} have constant row sums for $1 \le j \le n$. In addition, the row sums s_{1j} of Q_{1j} satisfy

$$[s_{11}, s_{12}, \dots, s_{1n}]^T = (R^T)^{-1}[1, 0, \dots, 0]^T.$$

Using a similar calculation, it is easy to check that all the Q_{ij} have constant row sums for $1 \le i, j \le n$, and $S^T = R^{-T}I$; that is, $SR = I$. □

Now we are ready to prove Theorem 1.

Proof of Theorem 1. Let $Q = (Q_{ij})_{N\times N}$ be the inverse of $\bar{G}$. Since the $\bar{G}_{ij}$, $i \ne j$, have constant row sums and the row sums of $\bar{G}$ are -1, it follows from Lemma 1 that the Q_{ij} have constant row sums for $1 \le i, j \le n$. Denote the row sum of $\bar{G}_{ij}$ by r_{ij} and that of Q_{ij} by s_{ij}. Then again from Lemma 1, we know that $S = R^{-1}$, where $R = (r_{ij})_{n\times n}$, and $S = (s_{ij})_{n\times n}$. So all the agents in the ith cluster have the same asymptotic value $-\sum_{j=1}^{n} s_{ij}a_j$.

Next we show that all the a_i that do not lead to n-cluster synchronization come from a set which has zero Lebesgue measure. Let $\mathcal{S} = \{x = [x_1, \dots, x_n]^T \in \mathbb{R}^n\colon x_i = x_j$ for some $i \ne j$ with $1 \le i, j \le n\}$, and let the smooth linear map $g : \mathbb{R}^n \to \mathbb{R}^n$ be defined by $g(x) = Rx$. Then it is easy to check that $\mathcal{S}$ has zero Lebesgue measure; so does $g(\mathcal{S})$. Let $\mathcal{U} = \{a = [a_1, \dots, a_n]^T \in \mathbb{R}^n\colon a_i \ne a_j$ for $i \ne j$; $(R^{-1}a)_i = (R^{-1}a)_j$ for some $i \ne j$ and $1 \le i, j \le n\}$; one has $\mathcal{U} \subset g(\mathcal{S})$, which implies that $\mathcal{U}$ has zero Lebesgue measure. If $a \notin \mathcal{U}$, system (5) realizes n-cluster synchronization, which completes the proof. □

The condition given in Theorem 1 is a sufficient condition, and it may not be necessary when $n > 2$. However, for the special case when $n = 2$, the condition is also necessary, as shown in the following result.

Theorem 2. *System (5) under any pair of distinct forcings $a_1 \ne a_2$ achieves 2-cluster synchronization for any initial condition if and only if the block matrices G_{ij}, $1 \le i, j \le 2$, and $i \ne j$, have constant row sums.*

Proof (*Sufficiency*)**.** Let $Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{21} & Q_{22} \end{bmatrix}_{N\times N}$ be the inverse of $\bar{G}$. It follows from the fact that the $\bar{G}_{ij}$ have constant row sums r_{ij} and Lemma 1 that the Q_{ij} have constant row sums s_{ij} and $S = \begin{bmatrix} -\frac{r_{21}+1}{r_{12}+r_{21}+1} & -\frac{r_{12}}{r_{12}+r_{21}+1} \\ -\frac{r_{21}}{r_{12}+r_{21}+1} & -\frac{r_{12}+1}{r_{12}+r_{21}+1} \end{bmatrix}$. Thus solutions of system (5) converge to $x^* = -\bar{G}^{-1}\bar{a} = -\begin{bmatrix} (a_1s_{11} + a_2s_{12})\mathbf{1}_{l_1} \\ (a_1s_{21} + a_2s_{22})\mathbf{1}_{l_2} \end{bmatrix}$. It is easy to check that $-a_1s_{11} - a_2s_{12} \ne -a_1s_{21} - a_2s_{22}$, since $a_1 \ne a_2$. Thus 2-cluster synchronization has been realized.

(Necessity) Suppose that system (5) realizes 2-cluster synchronization with final values $\bar{x}_1$ and $\bar{x}_2$. Let $\mathcal{K} = \{k \in \mathcal{N}$; the final value of $x_k(t)$ is $\bar{x}_1\}$. We first show that every agent under the same constant forcing is in the same cluster. Suppose on the contrary that the ith and jth agents both under constant forcing a_1 have different final values $\bar{x}_1$ and $\bar{x}_2$; then one has

$$0 = -\bar{x}_1 + a_1 + \sum_{k\in\mathcal{N}/\mathcal{K},k\ne i} g_{ik}(\bar{x}_2 - \bar{x}_1),$$

$$0 = -\bar{x}_2 + a_1 + \sum_{k\in\mathcal{K},k\ne j} g_{jk}(\bar{x}_1 - \bar{x}_2).$$

It follows that $(\bar{x}_2 - \bar{x}_1)(1 + \sum_{k\in\mathcal{N}/\mathcal{K},k\ne i} g_{ik} + \sum_{k\in\mathcal{K},k\ne j} g_{jk}) = 0$, which contradicts $\bar{x}_2 - \bar{x}_1 \ne 0$ and $1 + \sum_{k\in\mathcal{N}/\mathcal{K},k\ne i} g_{ik} + \sum_{k\in\mathcal{K},k\ne j} g_{jk} > 0$.

From the proof of sufficiency, we find that the equilibrium of system (5) is $x^* = -\begin{bmatrix} a_1Q_{11}\mathbf{1}_{l_1} + a_2Q_{12}\mathbf{1}_{l_2} \\ a_1Q_{21}\mathbf{1}_{l_1} + a_2Q_{22}\mathbf{1}_{l_2} \end{bmatrix}$. Let the ith row sums of Q_{11} and Q_{12} be t_{i1} and t_{i2}, respectively. Then, for any $1 \le i, j \le l_1$, and $a_1 \ne a_2$, we have $-a_1t_{i1} - a_2t_{i2} = -a_1t_{j1} - a_2t_{j2}$. It follows that $t_{i1} = t_{j1}$ and $t_{i2} = t_{j2}$ for $1 \le i, j \le l_1$. Thus, Q_{11} and Q_{12} have constant row sums. Applying similar arguments to Q_{21} and Q_{22}, one can conclude that G_{12} and G_{21} have constant row sums in view of Lemma 1. □

In the next subsection, we consider the systems that consist of not only informed agents, but also naive agents.

3.2. Systems of informed and naive agents

Now, consider the system consisting of $n - 1$ clusters of informed agents and one cluster of naive agents, whose dynamics are described respectively by

$$\dot{x}_i(t) = -x_i(t) + a_i + \sum_{j=1}^{N} g_{ij}x_j(t), \quad 1 \le i \le N - l_n, \tag{7}$$

and

$$\dot{x}_i(t) = \sum_{j=1}^{N} g_{ij}x_j(t), \quad N - l_n + 1 \le i \le N. \tag{8}$$

The system dynamics can be written in a compact form:

$$\dot{x}(t) = \bar{G}x(t) + \bar{a}, \tag{9}$$

where

$$\bar{G} = \begin{bmatrix} G_{11} - I & \cdots & G_{1,n-1} & G_{1n} \\ \vdots & \ddots & \vdots & \vdots \\ G_{n-1,1} & \cdots & G_{n-1,n-1} - I & G_{n-1,n} \\ G_{n1} & \cdots & G_{n,n-1} & G_{nn} \end{bmatrix},$$

and $\bar{a} = [a_1\mathbf{1}_{l_1}^T, \dots, a_{n-1}\mathbf{1}_{l_{n-1}}^T, \mathbf{0}_{l_n}^T]^T$.

Lemma 2. *$\bar{G}$ is invertible if and only if, for any naive agent, there is a directed path from some informed agent.*

Proof. See Appendix A. □

In what follows, we assume that for any naive agent there is always a directed path from some informed agent. Similar to the system consisting of only informed agents, since $\bar{G}$ is invertible, the equilibrium x^* of system (9) is $x^* = -\bar{G}^{-1}\bar{a}$. Let $y(t) = x(t) - x^*$; then one has $\dot{y}(t) = \bar{G}y(t)$. It is obvious that $y(t) \to 0$ as $t \to \infty$. Thus x^* is a globally stable equilibrium of system (9).

In order to ensure that agents in the same cluster have the same final values, we require the following. Suppose that the $\bar{G}_{ij}$ have constant row sums r_{ij} for $i = 1, \ldots, n-1$, $j = 1, \ldots, n$, and that the ith row sums of $G_{n1}, \ldots, G_{n,n-1}$ are $m_i h_1, \ldots, m_i h_{n-1}$ for $1 \le i \le l_n$, where the m_i are positive constants. We require that there is at least one $h_i \neq 0$, $1 \le i \le n-1$. Without loss of generality, suppose that $h_1, \ldots, h_k \neq 0$, $1 \le k \le n-1$, and $h_{k+1} = \cdots = h_{n-1} = 0$; it is easy to see that the ith row sums of G_{nn} are $-m_i \sum_{j=1}^{n-1} h_j$. Expanding the equation $Q\bar{G} = I$, following a similar argument as in the proof of Lemma 1, one has

$$\left(\begin{bmatrix} r_{11} & \cdots & r_{n-1,1} & h_1 \\ r_{12} & \cdots & r_{n-1,2} & h_2 \\ \vdots & \vdots & \ddots & \vdots \\ r_{1n} & \cdots & r_{n-1,n} & -\sum_{j=1}^{n-1} h_j \end{bmatrix} \otimes I\right) \begin{bmatrix} Q_{11}\mathbf{1} \\ Q_{12}\mathbf{1} \\ \vdots \\ Q_{1n}\mathbf{m} \end{bmatrix} = \begin{bmatrix} \mathbf{1} \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix},$$

where $\mathbf{m} \triangleq [m_1, \ldots, m_{l_n}]^T$. Let

$$M = \begin{bmatrix} h_2 r_{11} - h_1 r_{12} & \cdots & h_2 r_{n-1,1} - h_1 r_{n-1,2} \\ \vdots & \cdots & \vdots \\ h_k r_{11} - h_1 r_{1k} & \ddots & h_k r_{n-1,1} - h_1 r_{n-1,k} \\ \vdots & \ddots & \vdots \\ r_{1,n-1} & \cdots & r_{n-1,n-1} \\ -1 & \cdots & -1 \end{bmatrix};$$

then we have

$$(M \otimes I) \begin{bmatrix} Q_{11}\mathbf{1} \\ Q_{12}\mathbf{1} \\ \vdots \\ Q_{1,n-1}\mathbf{1} \end{bmatrix} = [h_2\mathbf{1}^T, \ldots, h_k\mathbf{1}^T, \ldots, \mathbf{0}^T, \mathbf{1}^T]^T.$$

M is invertible, since $\bar{G}$ is. Then, we can conclude that the Q_{1j} have constant row sums for $1 \le j \le n-1$. In addition, the row sums s_{1j} of the Q_{1j} satisfy

$$M[s_{11}, s_{12}, \ldots, s_{1,n-1}]^T = [h_2, \ldots, h_k, \ldots, 0, 1]^T.$$

It is easy to check that the Q_{ij}, $1 \le i \le n$, $1 \le j \le n-1$, have constant row sums s_{ij},

$$\tilde{S} = \begin{bmatrix} s_{11} & \cdots & s_{1,n-1} \\ \vdots & \ddots & \vdots \\ s_{n-1,1} & \cdots & s_{n-1,n-1} \end{bmatrix} = \begin{bmatrix} h_2 \cdots h_k & \mathbf{0}^T & 1 \\ -h_1 I & O & \mathbf{1} \\ O & I & \mathbf{1} \end{bmatrix} M^{-T},$$

and $[s_{n1}, \ldots, s_{n,n-1}] = [0, \ldots, 0, 1]M^{-T}$.

So $\tilde{S}$ is invertible. For $1 \le i \le n-1$, $\sum_{j=1}^{n} r_{ij} = -1$, it is easy to show that $\sum_{j=1}^{n-1} s_{ij} = -1$, for $1 \le i \le n$. Moreover, for $1 \le i \le n-1$ and $1 \le k \le l_n$, one can derive from $\bar{G}Q = I$ that

$$m_k h_1 s_{1i} + \cdots + m_k h_{n-1} s_{n-1,i} - m_k \sum_{j=1}^{n-1} h_j s_{ni} = 0.$$

It follows that $s_{ni} = \frac{\sum_{k=1}^{n-1} h_k s_{ki}}{\sum_{j=1}^{n-1} h_j}$.

Suppose that $\bar{x}_1, \ldots, \bar{x}_n$ are the final values of the n clusters; then each cluster converges to $\bar{x}_i = -\sum_{j=1}^{n-1} s_{ij} a_j$. It follows that $[\bar{x}_1, \ldots, \bar{x}_{n-1}]^T = -\tilde{S}[a_1, \ldots, a_{n-1}]^T$. Since $\tilde{S}$ is invertible, using a similar argument as in the proof of Theorem 1, one can conclude that, for almost all a_i with $a_i \neq a_j$ for $i \neq j$, the final values of the informed agents in different clusters are distinct from one another. In addition,

$$\begin{aligned} \bar{x}_n &= -\sum_{t=1}^{n-1} s_{nt} a_t = -\sum_{t=1}^{n-1} \sum_{k=1}^{n-1} \frac{h_k s_{kt}}{\sum_{j=1}^{n-1} h_j} a_t \\ &= \sum_{k=1}^{n-1} \frac{h_k}{\sum_{j=1}^{n-1} h_j} \left(-\sum_{t=1}^{n-1} s_{kt} a_t\right) = \sum_{k=1}^{n-1} \frac{h_k \bar{x}_k}{\sum_{j=1}^{n-1} h_j}, \end{aligned} \tag{10}$$

which implies that the final values of the naive agents have to be a linear combination of the final values of the informed agents. The coefficients $h_k / \sum_{j=1}^{n-1} h_j$ are determined by the row sums of $G_{n1}, \ldots, G_{n,n-1}$. Note that these final values only depend on the row sums of the submatrices of $\bar{G}$, but not on the number of agents and the proportion of the informed agents in the system. Hence, we have proved the following theorem.

Theorem 3. *For system (9), if, for any naive agent there is a directed path from some informed agent, the G_{ij} have constant row sums r_{ij} for $i = 1, \ldots, n-1$, $j = 1, \ldots, n$, and the ith row sums of $G_{n1}, \ldots, G_{n,n-1}$ are $m_i h_1, \ldots, m_i h_{n-1}$ for some $m_i > 0$, $1 \le i \le l_n$, then, for any initial condition, the final values of the clusters of the informed agents are distinct from one another for almost all (in the sense of Lebesgue measure) a_i for $1 \le i \le n-1$ with $a_i \neq a_j$ for $i \neq j$, and the final values of the naive agents converge to a linear combination of the asymptotic values of the informed agents as defined in (10).*

Remark 2. In Lu, Liu, and Chen (2010), more general agent dynamics are considered. Consequently, besides the condition of constant row sums stipulated in Theorem 3, additional conditions have to be imposed to guarantee clustering. Since more restricted agent dynamics are considered here, the agents' final values can be predicted, whereas it is difficult to do so for the model considered in Lu et al. (2010).

In this section, we have considered the clustering behavior when the agents have different linear dynamics. In the next section, we consider more challenging scenarios, in which agents are governed by the same self-dynamics.

4. Clustering with identical self-dynamics

Now, we consider the case when all the agents have the same self-dynamics:

$$\dot{x}_i(t) = f(t, x_i(t)) + c\sum_{j=1}^{N} g_{ij}\Gamma x_j(t), \quad 1 \le i \le N, \tag{11}$$

where the notation is the same as in (1), and f is a continuous map that is globally Lipschitzian in x_i with Lipschitz constant K and $g_{ij} \ge 0$ for $i \neq j$. There are existing results discussing when clustering might appear in (11). Now we compare such results.

Let $\mathcal{X}$ denote the manifold $\{x = [x_1^T(t), \ldots, x_N^T(t)]^T\colon x_1(t) = \cdots = x_{l_1}(t), x_{l_1+1}(t) = \cdots = x_{l_1+l_2}(t), \ldots, x_{N-l_n+1}(t) = \cdots = x_N(t)\}$ corresponding to the n-cluster synchronization. The following result is from Lu et al. (2010).

Theorem 4 (Lu et al., 2010). *The manifold $\mathcal{X}$ is invariant if and only if the block matrices G_{ij} achieved by partitioning G into submatrices corresponding to the clusters have constant row sums.*

A sufficient condition for the same n-cluster synchronization manifold to be invariant is given in Pogromsky (2008); it can be stated as follows.

Theorem 5 (Pogromsky, 2008). *The manifold $\mathcal{X}$ is invariant if there is a solution X to the linear equations*

$$(I_N - \Pi)G = X(I_N - \Pi), \tag{12}$$

where Π is a permutation matrix such that $\mathcal{X} = \ker(I_{mN} - \Pi \otimes I_m)$.

We now prove that the conditions given in Theorems 4 and 5 are in fact equivalent.

Proposition 1. *The block matrices G_{ij} of G have constant row sums if and only if there exists a solution X to the linear equations (12), where Π is a permutation matrix satisfying $\mathcal{X} = \ker(I_{mN} - \Pi \otimes I_m)$.*

Proof. (Necessity) Since $\mathcal{X} = \ker(I_{mN} - \Pi \otimes I_m)$, $\Pi = \mathrm{diag}\{\Pi_1, \ldots, \Pi_n\}$, where the Π_i are permutation matrices with the same dimensions of G_{ii}. From (12), we have

$$(I - \Pi_i)G_{ij} = X_{ij}(I - \Pi_j), \quad 1 \le i, j \le n. \tag{13}$$

Since the G_{ij} have constant row sums, the row sums of $(I - \Pi_i)G_{ij}$ are zero. Suppose that X_{ij} is a $u \times v$ matrix. Let $G_{ij}^T(I - \Pi_i)^T = [\beta_1, \beta_2, \ldots, \beta_u]$ and $X_{ij}^T = [\alpha_1, \ldots, \alpha_u]$, where α_i and β_i, $1 \le i \le u$, are column vectors. Then (13) is equivalent to

$$(I - \Pi_j)^T \alpha_k = \beta_k, \quad 1 \le k \le u. \tag{14}$$

Since $\mathrm{rank}(I - \Pi_j)^T = \mathrm{rank}([(I - \Pi_j)^T\ \beta_k]) = v - 1$, there exist solutions to (14). Then there exists a solution X to (12).

(Sufficiency) Without loss of generality, suppose that the permutation matrix Π can be written as $\Pi = \mathrm{diag}\{\Pi_1, \ldots, \Pi_q, \underbrace{1, \ldots, 1}_{n-q}\}$, where the Π_k, $1 \le k \le q$, are permutation matrices with the diagonal elements being zero. Then we can partition the matrix G into $n \times n$ blocks with the dimensions of G_{kk}, $q + 1 \le k \le n$, all being one. Thus we only need to prove that the G_{ij}, $1 \le i, j \le q$, have constant row sums. Let $G_{ij} = [\theta_1, \ldots, \theta_u]^T$, where the θ_i are column vectors. From (13), it follows that

$$(I - \Pi_i)G_{ij} = [\theta_1 - \theta_{i_1}, \ldots, \theta_u - \theta_{i_u}]^T = X_{ij}(I - \Pi_j),$$

where $\{i_1, \ldots, i_u\}$ is a permutation of $\{1, \ldots, u\}$ determined by Π_i. The row sums of $X_{ij}(I - \Pi_j)$ are zero because of the zero row sums of $I - \Pi_j$. In addition, the diagonal entries of Π_i are zero, so the row sums of θ_i^T, $1 \le i \le u$, are the same; namely the G_{ij} have constant row sums. □

We have just compared different conditions on when $\mathcal{X}$ is invariant. To further guarantee clustering to take place, we now introduce coupling delay into the system.

4.1. Delay-induced cluster synchronization

In view of Theorem 4, in this subsection we assume that the G_{ij} have constant row sums r_{ij}, $1 \le i, j \le n$. We introduce a coupling delay to (11) as follows (Lu, Chen, & Chen, 2006; Oguchi, Nijmeijer, & Yamamoto, 2008):

$$\begin{aligned}\dot{x}_i(t) &= f(t, x_i(t)) + c\sum_{j=1, j\ne i}^{N} g_{ij}\Gamma(x_j(t-\tau) - x_i(t)) \\ &= f(t, x_i(t)) + c\sum_{j=1}^{N} g_{ij}\Gamma x_j(t-\tau) \\ &\quad + cd_i\Gamma(x_i(t-\tau) - x_i(t)),\end{aligned} \tag{15}$$

where the notation is the same as in (11), and in addition $d_i = \sum_{j=1, j\ne i}^{N} g_{ij}$ is the in-degree of the ith agent, and $\tau > 0$ denotes the time delay. The initial condition for (15) is given by $x_i(\theta) = \phi_i(\theta)$, for $1 \le i \le N$, $\theta \in [-\tau, 0]$, where $\phi_i(\theta) \in C([-\tau, 0], \mathbb{R}^m)$. Since f is a continuous map that is globally Lipschitzian in x_i, and the couplings among agents are linear, system (15) has a unique solution which exists for all $t \ge 0$ (Driver, 1977).

When the N coupled agents achieve complete synchronization, i.e., $x_1(t) = x_2(t) = \cdots = x_N(t) = s(t)$, we have the following synchronized state equation:

$$\dot{s}(t) = f(t, s(t)) + cd_i\Gamma(s(t-\tau) - s(t)), \quad i = 1, \ldots, N. \tag{16}$$

Then, when $s(t-\tau) \ne s(t)$, a necessary condition for the synchronization manifold to be invariant is that $d_1 = d_2 = \cdots = d_N$. When the N coupled agents achieve n-cluster synchronization, i.e., $x_i(t) = x_j(t) = s_{\hat{i}}(t)$ for $\hat{i} = \hat{j}$, and $s_i(t) \ne s_j(t)$ for $\hat{i} \ne \hat{j}$, we have

$$\begin{aligned}\dot{s}_{\hat{i}}(t) &= f(t, s_{\hat{i}}(t)) + c\sum_{j\ne i, j\in C_{\hat{i}}} g_{ij}\Gamma(s_{\hat{i}}(t-\tau) - s_{\hat{i}}(t)) \\ &\quad + c\sum_{k=1, k\ne\hat{i}}^{n} r_{\hat{i}k}\Gamma(s_k(t-\tau) - s_{\hat{i}}(t)).\end{aligned}$$

Then, a necessary condition for the cluster synchronization manifold to be invariant is that $d_i = d_j$ for $\hat{i} = \hat{j}$ and $d_i \ne d_j$ for $\hat{i} \ne \hat{j}$.

Let $D = \mathrm{diag}\{d_1, \ldots, d_N\}$. Assume that $\mathbb{G}(G)$ is strongly connected; then G is irreducible. Hence, zero is a simple eigenvalue of G associated with a positive left eigenvector $\xi = [\xi_1, \xi_2, \ldots, \xi_N]^T$. Define $E = \mathrm{diag}\{\xi_1, \ldots, \xi_N\}$.

Now, consider the ith agent. Define the average of the $\hat{i}$th cluster to be

$$\bar{x}_{\hat{i}}(t) = \frac{1}{\sum_{i\in C_{\hat{i}}} \xi_i}\sum_{i\in C_{\hat{i}}} \xi_i x_i(t),$$

and the difference between agent i's state and this average to be $e_i(t) = x_i(t) - \bar{x}_{\hat{i}}(t)$. Then

$$\begin{aligned}\dot{e}_i(t) &= \dot{x}_i(t) - \dot{\bar{x}}_{\hat{i}}(t) \\ &= f(x_i(t)) + c\sum_{j=1}^{N} g_{ij}\Gamma x_j(t-\tau) \\ &\quad + cd_i\Gamma(x_i(t-\tau) - x_i(t)) - \dot{\bar{x}}_{\hat{i}}, \quad i = 1, \ldots, N.\end{aligned} \tag{17}$$

Let $e_i(t) = [e_{i1}(t), e_{i2}(t), \ldots, e_{im}(t)]^T \in \mathbb{R}^m$, $e(t) = [e_1^T(t), \ldots, e_N^T(t)]^T$, $\tilde{e}_i(t) = [e_{1i}(t), e_{2i}(t), \ldots, e_{Ni}(t)]^T \in \mathbb{R}^N$ and $\tilde{e}(t) = [\tilde{e}_1^T(t), \ldots, \tilde{e}_m^T(t)]^T$. Then one can check that

$$\sum_{i\in C_{\hat{i}}} \xi_i e_i = \sum_{i\in C_{\hat{i}}} \xi_i x_i - \sum_{i\in C_{\hat{i}}} \xi_i \left(\frac{1}{\sum_{i\in C_{\hat{i}}} \xi_i}\right)\sum_{i\in C_{\hat{i}}} \xi_i x_i = 0.$$

Hence,

$$\sum_{i\in C_{\hat{i}}} \xi_i e_i^T \dot{\bar{x}}_{\hat{i}}(t) = 0, \qquad \sum_{i\in C_{\hat{i}}} \xi_i e_i^T f(t, \bar{x}_{\hat{i}}(t)) = 0,$$

$$\sum_{i\in C_{\hat{i}}} \xi_i e_i^T \left(cd_i\Gamma(\bar{x}_{\hat{i}}(t-\tau) - \bar{x}_{\hat{i}}(t))\right) = 0,$$

$$\sum_{i\in C_{\hat{i}}} \xi_i e_i^T \left(\sum_{k=1}^{n}\sum_{j\in C_k} g_{ij}\Gamma\bar{x}_k(t)\right) = 0.$$

Since $f(t,x)$ satisfies the Lipschitz condition (2), there must exist a diagonal matrix $\Delta = \mathrm{diag}\{\delta_1,\ldots,\delta_m\}$ such that

$$(x-y)^T(f(t,x)-f(t,y)-\Delta(x-y)) \le -\alpha(x-y)^T(x-y) \tag{18}$$

holds for some $\alpha > 0$, all $x, y \in \mathbb{R}^m$, and all $t \ge 0$. A simple choice of Δ is $(K+\alpha)I$, while, for a specific $f(t,x)$ of interest, a less conservative Δ can be found. Now, we present the main result in this subsection.

Theorem 6. *Suppose that the G_{ij} have constant row sums r_{ij}, for $i, j = 1,\ldots,n$, that the in-degree d_i of each agent satisfies $d_i = d_j$ for $\hat{i} = \hat{j}$ and $d_i \ne d_j$ for $\hat{i} \ne \hat{j}$, and that Δ is a diagonal matrix satisfying (18). If there exist positive definite matrices Q_j such that the linear matrix inequalities*

$$\begin{bmatrix} 2\delta_j E - 2c\gamma_j ED + Q_j & c\gamma_j E(G+D) \\ c\gamma_j(G^T+D)E & -Q_j \end{bmatrix} < 0 \tag{19}$$

hold for all $j = 1,\ldots,m$, then $e_i(t) \to 0$ as $t \to \infty$ for $i = 1,\ldots,N$.

Proof. Since the matrix inequalities (19) are valid, there exists a positive constant ϵ such that $-2\alpha+\epsilon < 0$ and

$$\Lambda_j = \begin{bmatrix} 2\delta_j E - 2c\gamma_j ED + Q_j e^{\epsilon\tau} & c\gamma_j E(G+D) \\ c\gamma_j(G^T+D)E & -Q_j \end{bmatrix} < 0$$

hold for all $j = 1,\ldots,m$. Let

$$V_1 = \sum_{\hat{i}=1}^{n} W_{\hat{i}}(t) = \sum_{\hat{i}=1}^{n}\sum_{i\in C_{\hat{i}}} \xi_i e_i^T(t)e_i(t)e^{\epsilon t},$$

$$V_2 = \sum_{j=1}^{m}\int_{t-\tau}^{t} \tilde{e}_j^T(s)Q_j\tilde{e}_j(s)e^{\epsilon(s+\tau)}\mathrm{d}s.$$

Consider the candidate Lyapunov function $V(t) = V_1(t)+V_2(t)$. Then, for $W_{\hat{i}}(t) = \sum_{i\in C_{\hat{i}}}\xi_i e_i^T(t)e_i(t)e^{\epsilon t}$, its derivative along the solutions to (17) is

$$\begin{aligned}
\dot{W}_{\hat{i}} &= 2e^{\epsilon t}\sum_{i\in C_{\hat{i}}}\xi_i e_i^T(t)\Big(f(t,x_i(t)) - f(t,\bar{x}_{\hat{i}}(t)) + f(t,\bar{x}_{\hat{i}}(t)) \\
&\quad - \Delta e_i(t) + \Delta e_i(t) + c\sum_{j=1}^{N} g_{ij}\Gamma(x_j(t-\tau) - \bar{x}_{\hat{j}}(t-\tau)) \\
&\quad + c\sum_{k=1}^{n}\sum_{j\in C_k} g_{ij}\Gamma\bar{x}_k(t-\tau) \\
&\quad + cd_i\Gamma(x_i(t-\tau)-x_i(t)) - cd_i\Gamma(\bar{x}_{\hat{i}}(t-\tau)-\bar{x}_{\hat{i}}(t)) \\
&\quad + cd_i\Gamma(\bar{x}_{\hat{i}}(t-\tau)-\bar{x}_{\hat{i}}(t)) - \dot{\bar{x}}_{\hat{i}}(t)\Big) \\
&\quad + \epsilon e^{\epsilon t}\sum_{i\in C_{\hat{i}}}\xi_i e_i^T(t)e_i(t) \\
&= 2e^{\epsilon t}\sum_{i\in C_{\hat{i}}}\xi_i e_i^T(t)\Big(f(t,x_i(t)) - f(t,\bar{x}_{\hat{i}}(t)) \\
&\quad - \Delta e_i(t) + \Delta e_i(t) + c\sum_{j=1}^{N} g_{ij}\Gamma e_j(t-\tau) \\
&\quad + cd_i\Gamma(e_i(t-\tau)-e_i(t))\Big) + \epsilon e^{\epsilon t}\sum_{i\in C_{\hat{i}}}\xi_i e_i^T(t)e_i(t) \\
&\le (-2\alpha+\epsilon)e^{\epsilon t}\sum_{i\in C_{\hat{i}}}\xi_i e_i^T(t)e_i(t) \\
&\quad + 2e^{\epsilon t}\sum_{i\in C_{\hat{i}}}\xi_i e_i^T(t)\Big(\Delta e_i(t) + c\sum_{j=1}^{N} g_{ij}\Gamma e_j(t-\tau) \\
&\quad + cd_i\Gamma(e_i(t-\tau)-e_i(t))\Big).
\end{aligned}$$

Then, it follows that

$$\begin{aligned}
\dot{V}(t) &\le (-2\alpha+\epsilon)e^{\epsilon t}\sum_{\hat{i}=1}^{n}\sum_{i\in C_{\hat{i}}}\xi_i e_i^T(t)e_i(t) \\
&\quad + 2e^{\epsilon t}\sum_{\hat{i}=1}^{n}\sum_{i\in C_{\hat{i}}}\xi_i e_i^T(t)\Big(\Delta e_i(t) \\
&\quad + c\sum_{j=1}^{N} g_{ij}\Gamma e_j(t-\tau) + cd_i\Gamma(e_i(t-\tau)-e_i(t))\Big) \\
&\quad + e^{\epsilon(t+\tau)}\sum_{j=1}^{m}\tilde{e}_j^T(t)Q_j\tilde{e}_j(t) \\
&\quad - e^{\epsilon t}\sum_{j=1}^{m}\tilde{e}_j^T(t-\tau)Q_j\tilde{e}_j(t-\tau) \\
&\le (-2\alpha+\epsilon)e^{\epsilon t}\sum_{\hat{i}=1}^{n}\sum_{i\in C_{\hat{i}}}\xi_i e_i^T(t)e_i(t) \\
&\quad - e^{\epsilon t}\sum_{j=1}^{m}\tilde{e}_j^T(t-\tau)Q_j\tilde{e}_j(t-\tau) \\
&\quad + e^{\epsilon t}\sum_{j=1}^{m}\tilde{e}_j^T(t)\Big((2\delta_j E - 2c\gamma_j ED + Q_j e^{\epsilon\tau})\tilde{e}_j(t) \\
&\quad + 2c\gamma_j E(G+D)\tilde{e}_j(t-\tau)\Big) \\
&= (-2\alpha+\epsilon)e^{\epsilon t}\sum_{\hat{i}=1}^{n}\sum_{i\in C_{\hat{i}}}\xi_i e_i^T(t)e_i(t) \\
&\quad + e^{\epsilon t}\sum_{j=1}^{m}[\tilde{e}_j^T(t), \tilde{e}_j^T(t-\tau)]\Lambda_j\begin{bmatrix}\tilde{e}_j(t) \\ \tilde{e}_j(t-\tau)\end{bmatrix} \\
&\le 0.
\end{aligned}$$

Therefore, $V(t) \le V(0)$, which implies that $V_1(t)$ is bounded. In view of the definition of V_1, this further implies that $\|e(t)\|$ is bounded from above by an exponentially decaying signal that converges to zero. This completes the proof. □

Theorem 6 has shown that the differences among the states of the agents in the same cluster will converge to zero as time goes to infinity. However, it is in general difficult to prove that the differences between clusters do not converge to zero. Next, we prove 2-cluster synchronization when f is periodic. Consider

$$f(t,x_i(t)) = Bx_i(t) + h(x_i(t)) + \beta(t), \tag{20}$$

where $B = \mathrm{diag}\{b_1,\ldots,b_m\}$ with negative constants $b_i < 0$, $\beta : \mathbb{R}^+ \to \mathbb{R}^m$ is a continuous, periodic function with period $\omega > 0$, i.e., $\beta(t+\omega) = \beta(t)$, and $h : \mathbb{R}^m \to \mathbb{R}^m$ is a bounded function which satisfies $\|h(\xi_1)-h(\xi_2)\| \le H\|\xi_1-\xi_2\|$. We first present the following result.

Lemma 3. *If there exist positive definite matrices P_j such that the linear matrix inequalities*

$$\begin{bmatrix} 2(b_j+H)I - 2c\gamma_j D + P_j & c\gamma_j(G+D) \\ c\gamma_j(G^T+D) & -P_j \end{bmatrix} < 0 \tag{21}$$

hold for all $j = 1, \ldots, m$, then the coupled system (15) with f in the form of (20) has exactly one periodic solution with period ω to which all the other solutions converge exponentially fast as $t \to \infty$.

Proof. See Appendix B. $\square$

With Lemma 3, we now prove 2-cluster synchronization.

Theorem 7. *Suppose that the G_{ij} have constant row sums r_{ij}, for $i, j = 1, \ldots, n$, $\tau \neq k\omega$, for $k \geq 0$, and $d_i = d_j$ for $\hat{i} = \hat{j}$ and $d_i \neq d_j$ for $\hat{i} \neq \hat{j}$. If there exist positive definite matrices P_j and Q_j such that (19) and (21) hold with $\delta_j = b_j + H$ for $j = 1, \ldots, m$, then, for any initial condition, the coupled system (15) with f in the form of (20) realizes 2-cluster synchronization.*

Proof. In view of Theorem 6, we only need to show that complete synchronization cannot be achieved. Suppose that the contrary is true. Then (16) holds for all $i = 1, \ldots, N$. It follows from Lemma 3 that $s(t)$ is a periodic function with period ω. Since $\tau \neq k\omega$ for $k \geq 0$, it follows that $s(t-\tau)$ cannot be equal to $s(t)$ for all t. Thus we have $d_1 = d_N$, which contradicts the fact that $d_1 \neq d_N$, since agents 1 and N do not belong to the same cluster. $\square$

When $n \geq 3$, we show through simulations in Section 5 that n-cluster synchronization can be achieved if (19) and (21) are satisfied.

In the next subsection, we discuss a different mechanism to realize cluster synchronization when the agents' self-dynamics are identical.

4.2. Clustering with negative couplings

In this subsection, we study how clustering may appear as a pure effect of structured diffusive couplings. We assume that the agents' dynamics are completely determined by their couplings:

$$\dot{x}_i(t) = \sum_{j=1}^{N} g_{ij} x_j(t), \tag{22}$$

or, in a compact form,

$$\dot{x}(t) = Gx(t). \tag{23}$$

Comparing to (1), we have taken Γ to be an identity matrix, $c = 1$, and $m = 1$. The results derived below can be easily extended to the general case when $c > 0$ and $m \geq 1$. It is well known that, if $G_{ij} \geq 0, i \neq j$, and $\mathbb{G}(G)$ contains a directed spanning tree, then the system achieves consensus (Ren & Beard, 2005). In this subsection, we assume that there might be negative couplings, and as a result $G_{ij} \in \mathbb{R}$. Let $\eta_1 = [\mathbf{1}_{l_1}^T, \mathbf{0}_{N-l_1}^T]^T$, $\eta_2 = [\mathbf{0}_{l_1}^T, \mathbf{1}_{l_2}^T, \mathbf{0}_{N-l_1-l_2}^T]^T, \ldots, \eta_n = [\mathbf{0}_{N-l_n}^T, \mathbf{1}_{l_n}^T]^T$, and let $\alpha_1, \ldots, \alpha_n$ be n independent vectors satisfying $\eta_i^T \alpha_j = 1$, if $i = j$, and $\eta_i^T \alpha_j = 0$, if $i \neq j$. Since the solution to (23) is $x(t) = e^{Gt}x(0)$, it is obvious that, if

$$\lim_{t\to\infty} e^{Gt} = \sum_{i=1}^{n} \eta_i \alpha_i^T, \tag{24}$$

then n-cluster synchronization might be achieved. We provide the following necessary and sufficient condition under which (24) holds.

Lemma 4. *Eq. (24) holds if and only if*

$$G\eta_i = 0, \qquad \alpha_i^T G = 0, \quad i = 1, \ldots, n, \tag{25}$$

where G has exactly n zero eigenvalues and all the other eigenvalues have negative real parts.

Proof. See Appendix C. $\square$

From Lemma 4, it is clear that, in order to realize n-cluster synchronization, the G_{ij} have to have zero row sums. In the following discussion, assume that G satisfies the condition that the row sums of the G_{ij}, $1 \leq i, j \leq n$, are zero; then G has zero as an eigenvalue whose geometric multiplicity is at least n. Let $\eta_1 = [\mathbf{1}_{l_1}^T, \mathbf{0}_{N-l_1}^T]^T, \ldots, \eta_n = [\mathbf{0}_{N-l_n}^T, \mathbf{1}_{l_n}^T]^T$, be n right eigenvectors associated with 0, and let $\alpha_1, \ldots, \alpha_n$ be the corresponding left eigenvectors satisfying $\eta_i^T \alpha_j = 1$, if $i = j$, and $\eta_i^T \alpha_j = 0$, if $i \neq j$. The following result is a slightly modified version of the main result of Yu and Wang (2009).

Theorem 8. *Suppose that the initial values of system (23) satisfy that the $\alpha_i^T x(0)$ with $1 \leq i \leq n$ are not equal to each other; then n-cluster synchronization can be achieved if and only if G has exactly n zero eigenvalues and all the other eigenvalues have negative real parts.*

The conditions stipulated in Theorem 8 for achieving n-cluster synchronization is an algebraic condition, which is difficult to check in application. Now, we develop algorithms to construct appropriate coupling topologies which satisfy the conditions in Theorem 8.

Lemma 5 (*Horn & Johnson, 1985*)**.** *Let A and B be $N \times N$ Hermitian matrices, and let the eigenvalues $\lambda_i(A)$, $\lambda_i(B)$, and $\lambda_i(A+B)$ be arranged in decreasing order as $\lambda_N(\cdot) \leq \lambda_{N-1}(\cdot) \leq \cdots \leq \lambda_1(\cdot)$. For each $k = 1, 2, \ldots, N$, we have*

$$\lambda_k(A) + \lambda_N(B) \leq \lambda_k(A+B) \leq \lambda_k(A) + \lambda_1(B).$$

Intuitively, if the inner couplings within the clusters are strong enough, system (23) can achieve cluster synchronization. This is verified by the following results.

Proposition 2. *Let*

$$G = \mathrm{diag}\{c_1 G_{11}, \ldots, c_n G_{nn}\} + \begin{bmatrix} 0 & G_{12} & \cdots & G_{1n} \\ G_{21} & 0 & \cdots & G_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ G_{n1} & G_{n2} & \cdots & 0 \end{bmatrix}$$

be a symmetric matrix, $G_1 = \mathrm{diag}\{c_1 G_{11}, \ldots, c_n G_{nn}\}$, and $G_2 = G - G_1$. Suppose that the G_{ij} have zero row sums, matrices G_{ii} are irreducible, and the off-diagonal elements of G_{ii} are non-negative. If $c_i > \frac{\rho(G_2)}{-\max_{1\leq i\leq n}\lambda_2(G_{ii})}$, then G has exactly n zero eigenvalues and all the other eigenvalues are negative.

Proof. Since the G_{ij} have zero row sums, G has at least n zero eigenvalues. Using Lemma 5, one has

$$\lambda_N(G_2) \leq \lambda_i(G) - \lambda_i(G_1) \leq \lambda_1(G_2),$$

which leads to $|\lambda_i(G) - \lambda_i(G_1)| \leq \rho(G_2)$. It follows from $c_i > \frac{\rho(G_2)}{-\max_{1\leq i\leq n}\lambda_2(G_{ii})}$ that $\max_{1\leq i\leq n} c_i \lambda_2(G_{ii}) + \rho(G_2) < 0$. From the assumptions, one has that the $-G_{ii}$ are irreducible Laplacian matrices. It follows that $\lambda_1(G_1) = \cdots = \lambda_n(G_1) = 0$, and $\lambda_{n+1}(G_1) = \max_{1\leq i\leq n} c_i \lambda_2(G_{ii})$. Thus one concludes that $\lambda_{n+1}(G) \leq \max_{1\leq i\leq n} c_i \lambda_2(G_{ii}) + \rho(G_2) < 0$. $\square$

Proposition 3. *Suppose that the graphs $\mathbb{G}_1, \ldots, \mathbb{G}_n$ associated with $G_1, \ldots, G_n$ are balanced and strongly connected; then, for any positive definite matrix S with proper dimension, zero is an eigenvalue of $S\,\mathrm{diag}\{G_1, \ldots, G_n\}$ of algebraic and geometric multiplicity n, and all the other eigenvalues of $S\,\mathrm{diag}\{G_1, \ldots, G_n\}$ have negative real parts.*

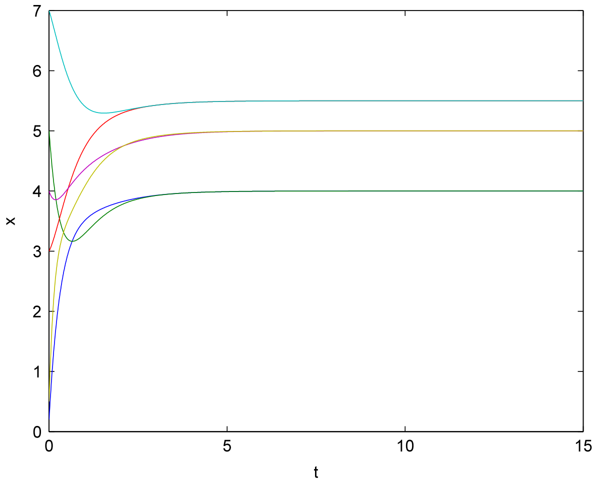


Fig. 1. The evolution of a system consisting of three clusters.

Proposition 3 can be proved using a similar argument as that in the proof of Theorem 4.5 in Lin (2008).

Proposition 3 provides a way to construct a graph satisfying the condition in Theorem 8. Let $\mathbb{G}'$ be a graph with n disconnected components, which are strongly connected and balanced. Let the matrix associated with $\mathbb{G}'$ be G'; then, multiplying from the left, a positive definite matrix S gives us a matrix $G = SG'$ satisfying the condition in Theorem 8.[2]

5. Illustrative examples

In this section, several examples are given to illustrate the theoretical analysis results.

Example 1. Consider the network consisting of two clusters of informed agents and one cluster of naive agents with $l_1 = l_2 = l_3 = 2$ and $a_1 = 1$, $a_2 = 7$. The coupling matrix is given by

$$G = \left[\begin{array}{cc:cc:cc} -2 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & 2 & 0 & 0 & 0 \\ \hdashline 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 \\ \hdashline 1 & 0 & 1 & 1 & -3 & 0 \\ 0 & 2 & 4 & 0 & 0 & -6 \end{array}\right].$$

Since the final values of the first and second clusters are 4 and 5.5, respectively, the values of the naive agents converge to $4 \times \frac{1}{3} + 5.5 \times \frac{2}{3} = 5$. Fig. 1 shows the evolution of the three clusters.

Example 2. Let

$$G_1 = \left[\begin{array}{cc:cc} -3 & 2 & 0 & 1 \\ 2 & -3 & 1 & 0 \\ \hdashline 1 & 0 & -2 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right],$$

$$G_2 = \left[\begin{array}{cc:cc:cc} -4 & 2 & 0 & 1 & 0 & 1 \\ 2 & -4 & 1 & 0 & 1 & 0 \\ \hdashline 1 & 0 & -3 & 1 & 1 & 0 \\ 0 & 1 & 1 & -3 & 0 & 1 \\ \hdashline 0 & 1 & 1 & 0 & -2 & 0 \\ 1 & 0 & 0 & 1 & 0 & -2 \end{array}\right].$$

All the agents in the diffusively coupled network (15) have the same self-dynamics, which are (Zhou, Xiang, & Liu, 2007)

$$\dot{x}_i(t) = \begin{bmatrix} -3.6 & 0 \\ 0 & -4.2 \end{bmatrix} \begin{bmatrix} x_{i1}(t) \\ x_{i2}(t) \end{bmatrix} + \begin{bmatrix} a\cos(\nu t) \\ 0 \end{bmatrix} + \begin{bmatrix} 1.5 & -0.5 \\ -2.1 & 1.8 \end{bmatrix} \begin{bmatrix} \frac{1}{2}(|x_{i1}(t)+1| - |x_{i1}-1|) \\ \frac{1}{2}(|x_{i2}(t)+1| - |x_{i2}-1|) \end{bmatrix}. \quad (26)$$

When $a = 1.6$ and $\nu = 2.6$, system (26) has a unique and globally exponentially stable periodic solution.

Consider the coupled network associated with the coupling matrix G_1. Let $\tau = 1$, $c = 0.5$, and $\Gamma = \mathrm{diag}\{1, 1\}$. Using Matlab, we get solutions Q_j and P_j to (19) and (21) as $Q_j = P_j = \mathrm{diag}\{0.5550, 0.5550, 0.4717, 0.4717\}$, $j = 1, \ldots, m$. Assume that every agent takes the same initial value $x_i(\theta) = [0.1, 0.2]^T$, $i = 1, \ldots, 4$, $\theta \in [-1, 0]$. The states of the agents finally evolve into two clusters, as shown in Fig. 2(a). When $\tau = 0$, the states of the agents achieve complete synchronization, as shown in Fig. 2(b). So the delay indeed has induced the clustering behavior in this example.

When the coupled network has the coupling matrix G_2, and $\tau = 1$, from Fig. 3 it can be seen that the agents finally evolve into three clusters.

Example 3. A network that realizes 2-cluster synchronization has the topology shown in Fig. 4. The associated matrix G is

$$\left[\begin{array}{cccc:cc} -2 & 2 & 0 & 0 & -1 & 1 \\ 2 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 2 & 1 & -1 \\ 0 & 0 & 2 & -2 & 0 & 0 \\ \hdashline 0 & 1 & 0 & -1 & -2 & 2 \\ 0 & -1 & 0 & 1 & 2 & -2 \end{array}\right],$$

which has two zero eigenvalues and the other eigenvalues have negative real parts. Let groups 1, 2, 3 be $\{1, 2\}$, $\{3, 4\}$, $\{5, 6\}$, respectively. It is easy to see from Figs. 4 and 5 that, although there is no direct connection between groups 1 and 2, the states of the agents in these two groups finally achieve the same value via the interconnection with agents in group 3, which have a different final value.

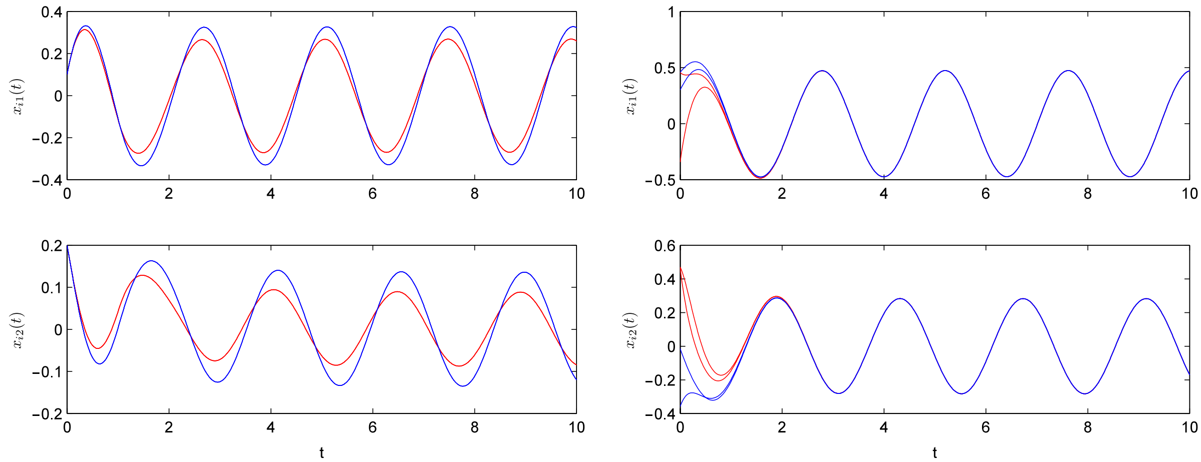
6. Concluding remarks

This paper has investigated three different mechanisms that lead to n-cluster synchronization in multi-agent systems. Some sufficient conditions and/or necessary conditions have been constructed for systems with different agent self-dynamics, with delay, or having negative couplings. Numerical examples are given to verify the effectiveness of the analysis. The three mechanisms presented in this paper are just examples of different approaches towards cluster synchronization. It is envisioned that after gaining insights into the clustering behavior in natural, social, or engineered systems, more mechanisms can be revealed, and thus different cluster synchronization models can be constructed whose advantages and disadvantages can be compared. The mechanisms studied in this paper might lead to new understanding of the clustering behavior in natural and man-made systems, and in the end help to design efficient coordination algorithms for dynamic multi-agent systems.

Appendix A. Proof for Lemma 2

(Sufficiency) Assume that $|\bar{G}| = 0$. Then $\bar{G}x = 0$ has a non-trivial solution $x_1, \ldots, x_N$. Let r be one of the indices for which $|x_i|$, $i = 1, \ldots, N$, is maximum. Then $|x_i| < |x_r|$, for $1 \le i \le l_1 + \cdots + l_{n-1}$.

[2] We are indebted to I. Shames for pointing out this reformulation of some of our earlier results.



(a) When $\tau = 1$, the agents evolve into two clusters.

(b) When $\tau = 0$, the agents achieve complete synchronization.

Fig. 2. The evolution of the states $x_i(t)$ for $i = 1, \ldots, 4$.

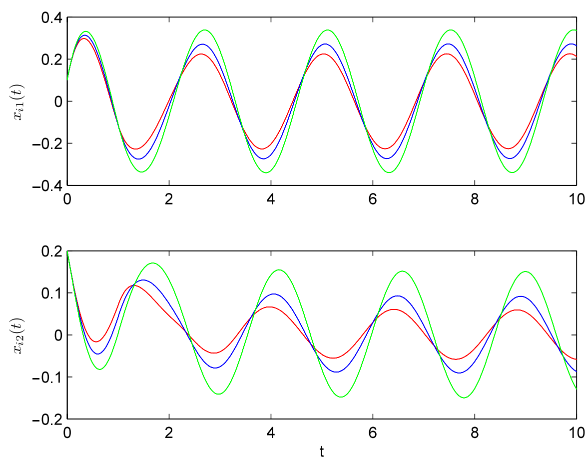


Fig. 3. The agents evolve into three clusters with G_2 when $\tau = 1$.

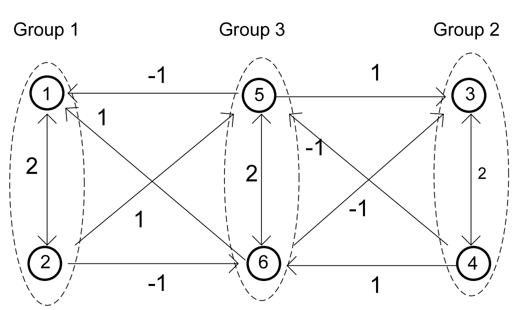


Fig. 4. The topology of a network.

Suppose that the contrary is true. Then consider the ith row of $\bar{G}x$. One has

$$(-g_{ii} + 1)|x_r| = (-g_{ii} + 1)|x_i| \le \sum_{j\neq i} g_{ij}|x_j| \le \sum_{j\neq i} g_{ij}|x_r|.$$

It follows that $|x_r| \le 0$, which contradicts the fact that $|x_r| > 0$. We conclude that $r > l_1 + \cdots + l_{n-1}$.

For any k satisfying $|x_r| > |x_k|$, one has $g_{rk} = 0$. Otherwise, consider the rth row of $\bar{G}x$; one has

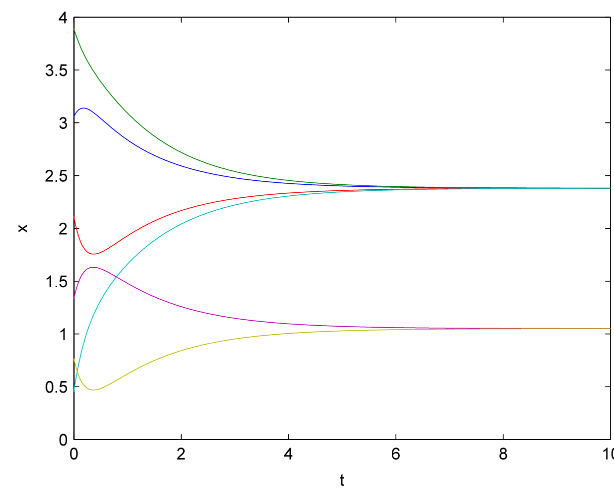


Fig. 5. State trajectories. (Agents 1, 2, 3, 4 are in the same cluster.)

$$-g_{rr}|x_r| \le \sum_{j\neq r} g_{rj}|x_j| < \sum_{j\neq r} g_{rj}|x_r| = -g_{rr}|x_r|,$$

which is a contradiction.

Let s be the number of indices j for which $|x_j| = |x_r|$. Then the rth row contains $N - s$ zeros and $g_{rk} = 0$, for $1 \le k \le l_1 + \cdots + l_{n-1}$. All the s corresponding rows contain $N - s$ zeros in the same places. So, by the same permutations of the rows and columns, matrix $\bar{G}$ can be transformed to

$$\begin{bmatrix} U_{11} & U_{12} \\ 0 & U_{22} \end{bmatrix}, \tag{A.1}$$

where $U_{22} \in \mathbb{R}^{s\times s}$ is a square matrix and U_{11} contains $\begin{bmatrix} G_{11} - I & \cdots & G_{1,n-1} \\ \vdots & \ddots & \vdots \\ G_{n-1,1} & \cdots & G_{n-1,n-1} - I \end{bmatrix}$ as a submatrix in the upper left corner. Thus there is no directed path from any informed agent to the naive agent in the block U_{22}.

(Necessity) If, for s naive agents, there are no directed paths from any informed agent, then $\bar{G}$ can be transformed to (A.1) by the same permutations of the rows and columns such that U_{22} only contains s naive agents. U_{22} having zero row sum implies that $|\bar{G}| = 0$, which is a contradiction.

Appendix B. Proof for Lemma 3

Let $\mathcal{C} = C([-\tau, 0], \mathbb{R}^m)$. For any $\phi_i \in \mathcal{C}$, we define $\|\phi_i\|_\tau = \sup_{-\tau \le \theta \le 0} \|\phi_i(\theta)\|$. For any $\phi = [\phi_1^T, \ldots, \phi_N^T]^T$, where $\phi_i \in \mathcal{C}$, $1 \le i \le N$, we denote the solution of (15) through $(0, \phi)$ as $x(t, \phi) = [x_1^T(t, \phi), \ldots, x_N^T(t, \phi)]^T$, and define $x_t(\phi) = x(t+\theta, \phi)$, $\theta \in [-\tau, 0]$, $t \ge 0$; then $x_t(\phi) \in \mathcal{C}$ for all $t \ge 0$.

Now consider two solutions $x(t, \phi)$ and $x(t, \varphi)$ of (15). Define $w_i(t) = x_i(t, \phi) - x_i(t, \varphi)$, $w(t) = [w_1^T(t), \ldots, w_N^T(t)]^T$, $\tilde{w}_i(t) = [w_{1i}(t), \ldots, w_{Ni}(t)]^T$, and $\tilde{w}(t) = [\tilde{w}_1^T(t), \ldots, \tilde{w}_m^T(t)]^T$. It follows from (15) and (20) that

$$\dot{w}_i(t) = Bw_i(t) + h(x_i(t, \phi)) - h(x_i(t, \varphi)) + c\sum_{j=1}^{N} g_{ij}\Gamma w_j(t-\tau) + cd_i\Gamma(w_i(t-\tau) - w_i(t)).$$

Since the matrix inequalities (21) are valid, there exists a positive constant ϵ such that

$$\Omega_j = \begin{bmatrix} 2(b_j + H)I + \epsilon I - 2c\gamma_j D + P_j e^{\epsilon\tau} & c\gamma_j(G + D) \\ c\gamma_j(G^T + D) & -P_j \end{bmatrix}$$

are negative definite for all $j = 1, \ldots, m$. Consider the candidate Lyapunov function

$$V(t) = \sum_{i=1}^{N} w_i^T(t)w_i(t)e^{\epsilon t} + \sum_{j=1}^{m}\int_{t-\tau}^{t} \tilde{w}_j^T(s)P_j\tilde{w}_j(s)e^{\epsilon(s+\tau)}\mathrm{d}s.$$

By similar calculations as in the proof of Theorem 6, we obtain

$$\dot{V}(t) \le e^{\epsilon t}\sum_{j=1}^{m}[\tilde{w}_j^T(t), \tilde{w}_j^T(t-\tau)]\Omega_j \begin{bmatrix} \tilde{w}_j(t) \\ \tilde{w}_j(t-\tau) \end{bmatrix} \le 0.$$

Therefore, $V(t) \le V(0)$, from which it follows that

$$\|x(t, \phi) - x(t, \varphi)\| \le Me^{-\frac{\epsilon}{2}t}\|\phi - \varphi\|_\tau, \quad t \ge 0,$$

where $M \ge 1$ is a constant. Then, it is easy to see that

$$\|x_t(\phi) - x_t(\varphi)\|_\tau \le Me^{-\frac{\epsilon}{2}(t-\tau)}\|\phi - \varphi\|_\tau. \tag{B.1}$$

Comparing (B.1) and Eq. (5) in Cao (1999), it is easy to see that, using similar arguments to that in Cao (1999), one can conclude that system (15) has exactly one periodic solution with period ω, and all the other solutions converge exponentially to it as $t \to \infty$.

Appendix C. Proof for Lemma 4

We give the proof for the case when $n = 2$. The proof for the general case $n \ge 2$ can be proved following similar steps.

(Sufficiency) This has been proved as Lemma 6 in Yu and Wang (2009).

(Necessity) Let $J = \mathrm{diag}\{J_1, \ldots, J_s\}$ be the Jordan form of G, i.e., there exists a non-singular matrix P such that $G = PJP^{-1}$. Then

$$\lim_{t\to\infty} e^{Gt} = P\lim_{t\to\infty}\mathrm{diag}\{e^{J_1 t}, \ldots, e^{J_s t}\}P^{-1}.$$

$\lim_{t\to\infty} e^{Gt}$ exists if and only if the J_i are zero matrices or the eigenvalues of the J_i have negative real parts. Let $u_1, \ldots, u_N$ be the columns of P and $v_1^T, \ldots, v_N^T$ be the rows of P^{-1}. Then the fact that (24) holds implies that J has the form $J = \mathrm{diag}\{O_k, Z\}$, where O_k is the k-dimensional zero matrix and the eigenvalues of Z have negative real parts. We have

$$\lim_{t\to\infty} e^{Gt} = P\begin{bmatrix} I_k & 0 \\ 0 & 0 \end{bmatrix}P^{-1} = \sum_{i=1}^{k} u_i v_i^T.$$

Since $\mathrm{rank}(u_i v_i^T) = 1$ and $\mathrm{rank}(\sum_{i=1}^N u_i v_i^T = I) = N$, $\sum_{i=1}^k u_i v_i^T$ must have rank k. Combined with (24), one has $k = 2$ and $u_1 v_1^T + u_2 v_2^T = \eta_1\alpha_1^T + \eta_2\alpha_2^T$, which implies that G has exactly two zero eigenvalues and all the other eigenvalues have negative real parts. In addition, one has

$$u_{11}v_1^T + u_{21}v_2^T = \alpha_1^T, \ldots, u_{1l_1}v_1^T + u_{2l_1}v_2^T = \alpha_1^T,$$

which implies that $(u_{1i} - u_{1j})v_1^T + (u_{2i} - u_{2j})v_2^T = 0$. Then $u_{1i} = u_{1j}$ and $u_{2i} = u_{2j}$ for $1 \le i, j \le l_1$. Using similar arguments, we have

$$u_{11} = \cdots = u_{1l_1}, \qquad u_{1l_1+1} = \cdots = u_{1N},$$
$$u_{21} = \cdots = u_{2l_1}, \qquad u_{2l_1+1} = \cdots = u_{2N}.$$

If $u_{11} = 0$, then $[\mathbf{0}_{l_1}^T, \mathbf{1}_{l_2}^T]^T$ is a right eigenvector associated with 0, and so is $[\mathbf{1}_{l_1}^T, \mathbf{0}_{l_2}^T]^T$. If $u_{11} \ne 0$, $[\mathbf{0}_{l_1}^T, \frac{u_{2N}u_{11} - u_{1N}u_{21}}{u_{11}}\mathbf{1}_{l_2}^T]^T$ is a right eigenvector associated with 0. So $[\mathbf{0}_{l_1}^T, \mathbf{1}_{l_2}^T]^T$ and $[\mathbf{1}_{l_1}^T, \mathbf{0}_{l_2}^T]^T$ are right eigenvectors associated with 0.

Without loss of generality, choose $u_1 = \eta_1 = [\mathbf{1}_{l_1}^T, \mathbf{0}_{l_2}^T]^T$ and $u_2 = \eta_2 = [\mathbf{0}_{l_1}^T, \mathbf{1}_{l_2}^T]^T$; then $\eta_1(v_1 - \alpha_1)^T + \eta_2(v_2 - \alpha_2)^T = 0$, which implies that $v_1 = \alpha_1$ and $v_2 = \alpha_2$. Hence, one has $\alpha_1^T G = \alpha_2^T G = 0$.

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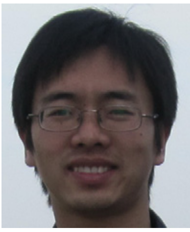

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