Recent developments in solving the generalized Collatz problem

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Consider a sequence of integers :

$x \longrightarrow \frac{x}{2}$ if x is even, $x \longrightarrow \frac{3x+1}{2}$ if x is odd.

What is the behavior of this sequence?

See for enormous literature Jeff Lagarias bibles
`http://arxiv.org/abs/math.NT/0309224`
`http://arxiv.org/abs/math.NT/0608208`

Paul Erdös : **Mathematics is not ready for such problems!**

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Experimental results

If $x = 17$ we find 17, 26, 13, 40, 20, 10, 5, 8, 4, 2, 1, 2, ...

The famous Collatz or $3x + 1$ conjecture states that for every $x > 0$ finally the cycle $(1, 2)$ occurs, however 27 increases to 9232 before returning to $(1, 2)$ and 562 380 758 422 254 271 $\sim 10^{18}$ increases above 10^{30} before returning to $(1, 2)$. For $x \leq 0$ other cycles exist !

The search for cycles goes on, see
`http://www.ericr.nl/wondrous/`
`http://www.ieeta.pt/~tos/3x+1.html`

The Collatz or $3x + 1$ problem is alive and unproved since 1930.

Steiner's method for cycle existence

Define an m-cycle as a hypothetical cycle with m local minima x_i. Steiner proved in 1978 that $(1,2)$ is the only 1-cycle if $x_0 > 0$. If $x_0 = a2^k - 1$ is odd then $x_k = a3^k - 1$ is even. If a 1-cycle exists then $\frac{a3^k-1}{2^l} = a2^k - 1$ from which follows $1 < \frac{2^{k+l}}{3^k} < \frac{a2^k}{a2^k-1}$ and

$$0 < \Lambda = (k+l)\log 2 - k\log 3 < \frac{1}{a2^k - 1} \leq \frac{1}{2^k - 1}. \qquad (1)$$

If $k \geq 7$ then $\frac{1}{2^k-1} < \frac{1}{2k^2}$ thus convergents of continued fractions only. Transcendence theory (Baker's lemma) gives a lower bound $\Lambda > e^{-C(\log k)^2}$. This leads to an upper bound for k and there appears to be no other solution than $k = l = 1$.

Non-existence of m-cycles for $m < 78$

(joint work with Benne de Weger)

Call the minima of an m-cycle $x_i = a_i 2^{k_i} - 1$ and the maxima $y_i = a_i 3^{k_i} - 1$ for $i = 0, \ldots m-1$. Instead of Steiner's equation for a 1-cycle $\frac{a3^k - 1}{2^l} = a2^k - 1$ we now have a chain equation $\frac{a_i 3^{k_i} - 1}{2^{l_i}} = a_{i+1} 2^{k_{i+1}} - 1$.

This results in a diophantine system in the coefficients a_i.

$$\begin{pmatrix} -3^{k_0} & 2^{k_1+l_0} & & \\ & -3^{k_1} & 2^{k_2+l_1} & \\ \cdots & & & \\ & & & \\ 2^{k_0+l_{m-1}} & & & -3^{k_{m-1}} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \cdots \\ \\ a_{m-1} \end{pmatrix} = \begin{pmatrix} 2^{l_0} - 1 \\ 2^{l_1} - 1 \\ \cdots \\ \\ 2^{l_{m-1}} - 1 \end{pmatrix} \tag{2}$$

S & de Weger's method (2)

Instead of Steiner's upper bound for Λ

$$0 < \Lambda = (k+l)\log 2 - k\log 3 < \frac{1}{a2^k - 1} \le \frac{1}{2^k - 1} \qquad (3)$$

we find by multiplication of all chain equations

$$0 < (K+L)\log 2 - K\log 3 < \sum_{i=0}^{m-1} \frac{1}{a_i 2^{k_i} - 1} \le \sum_{i=0}^{m-1} \frac{1}{2^{k_i} - 1}. \qquad (4)$$

To find an upper bound as a function of K we note that ($\delta = \log_2 3$) if $x_i \ge X_0$ then all these x_i are of the "same" size .

$$x_{i+1} = \frac{y_i}{2^{\ell_i}} < \frac{y_i + 1}{2^{\ell_i}} = \frac{3^{k_i} a_i}{2^{\ell_i}} \le \frac{3^{k_i}}{2} a_i = \frac{3^{k_i}}{2} \frac{x_i + 1}{2^{k_i}} = \left(\frac{3}{2}\right)^{k_i} \frac{x_i + 1}{2} = \frac{1}{2}\left(2^{k_i}\right)^{\delta - 1}(x_i + 1)$$
$$< \frac{1}{2}\left(2^{k_i} a_i\right)^{\delta - 1}(x_i + 1) = \frac{1}{2}(x_i + 1)^{\delta} \le \frac{1}{2}\left(1 + X_0^{-1}\right)^{\delta} x_i^{\delta} = b^{\delta} x_i^{\delta}.$$

S & de Weger's method (3)

This inequality holds cyclically. As a result all x_i can be estimated in terms of the minimal x_i, say x_0 and we find the K, m dependent upper bound

$$0 < \Lambda = (K+L)\log 2 - K \log 3 < m c_m 2^{-\frac{\delta - 1}{\delta^m - 1} K}. \tag{5}$$

where $c_m \sim 0.6$ depends marginally on X_0. Apart from new developments in continued fractions and transcendence theory e.g. Rhin's lemma

$$\Lambda > e^{-13.3(0.46057 + \log K)} \tag{6}$$

from here on the **Steiner proof continues if K_{min} from X_0 is large enough for continued fractions theory.**

Let $\frac{p_n}{q_n}$ be the n^{th} convergent to δ . If $q_n + q_{n+1} \leq (\log 2)\frac{X_0}{m}$ then $K \geq K_0(m) = q_{n+1}$. If not then $\Lambda = (\log 2) \mid (K+L) - K\delta \mid \geq (\log 2) \mid p_n - q_n \delta \mid > \frac{\log 2}{q_n + q_{n+1}} \geq \frac{m}{X_0}$

Generalizations for px+q sequences

For the $7x+5$ problem we find 1, 6, 3, 13, 48, 24, 12, 6, 3, ..., 5, 20, 10, 5, ..., (non primitive via $7x+1$) 7, 27, ... 3688, ... 27, ... and many divergent sequences.

As an exotic example consider the $97x+32641759$ problem. We find cycles of length 25 starting with $x_0 = 9607, 9611, 9619, 9635, \ldots$ and many divergent trajectories.

Can SdW method explain this?

SdW method requires
(1) a smart expression for x_0 leading to a diophantine system
(2) an upper bound for Λ in a_i, k_i
(3) an upper bound in K, m.

A smart expression for x_0 and an upper bound for $\Lambda(p,q)$

$x_0 = a2^k - 1$ does in general not lead to $x_k = ap^k - 1$, however if $(p-2)x_0 = a2^k - q$ then $(p-2)x_1 = ap2^{k-1} - q$ and $(p-2)x_k = ap^k - q$. This leads to the diophantine system

$$\begin{pmatrix} -p^{k_0} & 2^{k_1+l_0} & & \\ & -p^{k_1} & & \\ \dots & & & \\ & & & \\ 2^{k_0+l_{m-1}} & & & -p^{k_{m-1}} \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \dots \\ \\ a_{m-1} \end{pmatrix} = \begin{pmatrix} q(2^{l_0}-1) \\ q(2^{l_1}-1) \\ \dots \\ \\ q(2^{l_{m-1}}-1) \end{pmatrix} \tag{7}$$

This leads to $0 < \Lambda(p,q) = (K+L)\log 2 - K \log p < \sum_{i=0}^{m-1} \frac{q}{a_i 2^{k_i} - q}$

and ($\rho = \log_2 p$) if $x_i > X_0$ to

$$0 < (K+L)\log 2 - K\log p < [\frac{(p-1)^\rho}{2(p-2)}]^{\frac{1}{\rho-1}} \cdot 2^{\frac{\log_2(p-1)}{\rho^{m-1}}} \cdot \frac{qm}{p-2} \cdot 2^{-\frac{K}{m\rho^{m-1}}}$$

For each p,q all m-cycles up to a given upper bound for m can be found following S & de Weger.

Additional conditions for cycles of the $px+q$ problem

S [2006] accepted by Acta Arithmetica

Lemma 1 *A necessary and sufficient condition for the existence of a (non) primitive* 1*-cycle for the* $px+q$ *problem is the existence of positive integers* k, l *and* r*(odd) such that* $2^{k+l}-p^k=q\cdot r$ *and the existence of an odd integer* x_0 *such that* $x_0=\frac{p^k-2^k}{(p-2)r}$

1-cycle $\rightarrow a(2^{k+l}-p^k)=q(2^l-1)\rightarrow 2^{k+l}-p^k=qr$ further

$(p-2)x_0=a2^k-q=\frac{q(2^l-1)}{2^{k+l}-p^k}2^k-q=\frac{2^l-1}{r}2^k-q=\frac{2^{k+l}-2^k-qr}{r}=\frac{p^k-2^k}{r}$

Corollary 2 *If for the* $px+q$ *problem a solution exists of the equation* $2^{k+l}-p^k=q$*, then there exist a (non) primitive* 1*-cycle with* $x_0=\frac{p^k-2^k}{p-2}$*.*

Examples from corollary 2

Consider as an example the $11x+7$ problem. We have $2^7 - 11^2 = 7$, hence $k = 2$, $l = 5$ and $x_0 = \frac{(11^2-2^2)\cdot 7}{(11-2)\cdot 7} = 13$. We find the 1-cycle (13,75,416,208,104,52,26). We computed for primes $5 \leq p \leq 97$ the minimal values for q (prime) for which $2^{k+l} - p^k = q$ has a solution with $k \geq 2$.

p	q	p	q	p	q
5	3	31	1087	67	28279
7	79	37	6823	71	126031
11	7	41	367	73	125743
13	1879	43	199	79	1951
17	223	47	30559	83	1303
19	151	53	29959	89	271
23	4217	59	29287	97	32641759
29	7351	61	520567		

Minimal q with $2^{k+l} - p^k = q$.

A lower bound for the number of cycles

S submitted [2007]

Let $C(m,p,q)$ be the number of primitive m-cycles of the $px+q$ problem and let $C(p,q) = \sum_m C(m,p,q)$. Let $S(p)$ be the set of primes q such that $2^{k+l} - p^k = q$ has a solution with $k \geq 2$.

• For each p, there exist infinitely many $px+q$ problems with $C(1,p,q) \geq 1$.

• For each p and $d > 0$, there exist infinitely many $px+q$ problems with $C(p,q) > d$.

• If $S(p)$ is an infinite set (conjectured, no proof) and $q \in S(p)$ then for each $d > 0$ there exist infinitely many such $px+q$ problems with $C(p,q) > d$.

Exotic Collatz sequences with a large number of cycles $C(p,q)$

q	k	l	$C(5,q) \geq$
3	3	4	3
7	2	3	3
103	2	5	3
131	3	5	9
487	2	7	4
971	5	7	66
1423	4	7	37
8167	2	11	6
13259	5	9	173
32143	4	11	95
130447	4	13	140
259019	5	13	489

q	k	l	$C(7,q) \geq$
79	2	5	3
463	2	7	5
1999	2	9	5
5791	4	9	62
30367	4	11	95
32719	2	13	7
130729	3	14	40
131023	2	15	8
521887	4	15	206

Another exotic example is the $93x+32641759$ problem with 92 cycles with cycle length 25, starting with $9607, 9611, 9619, 9635, 9667, 9731, 9805, 9813\ldots$.