A simple (inductive) proof for the non-existence of 2-cycles of the $3x + 1$ problem

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Abstract

A 2-cycle of the $3x + 1$ problem has two local odd minima x_0 and x_1 with $x_i = a_i 2^{k_i} - 1$. Such a cycle exists if and only if an integer solution exists of a diophantine system of equations in the coefficients a_i. We derive a numerical lower bound for $a_0 \cdot a_1$, based on Steiner's proof for the non-existence of 1-cycles. We derive an analytical expression for an upper bound for $a_0 \cdot a_1$ as a function of K and L (the number of odd and even numbers in the cycle). We apply a result of de Weger on linear logarithmic forms to show that these lower and upper bound are contrary. The proof does not use exterior lower bounds for numbers in a cycle and for the cycle length.

1 Introduction

The $3x + 1$ problem is defined by a sequence of natural numbers, generated conditionally by $T(x) = \frac{1}{2}(3x+1)$ if x is odd and by $T(x) = \frac{1}{2}x$ if x is even. The problem has been analyzed from various viewpoints [4]. A famous conjecture states that for all natural numbers eventually the cycle $(1, 2)$ appears. A 1-cycle contains one increasing subsequence and one decreasing subsequence and has consequently exactly one minimum and one maximum. Steiner [9] proved that there are no other 1-cycles then $(1, 2)$.

A 2-cycle contains four (alternating increasing and decreasing) subsequences and has two local minima and two local maxima. We call (1,2,1,2) the trivial 2-cycle and any other 2-cycle non-trivial. Let K be the number of odd elements and L be the number of even elements in the cycle. Recently Simons [7] has derived necessary conditions for the existence of 2-cycles. Using transcendence theory [6] with an appropriate lower bound for numbers in a cycle ($x_i > 100$) and for the (odd numbers) cycle length ($K > 100$), he proved that non-trivial

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2-cycles do not exist. Simons and de Weger [8] proved the non-existence of non-trivial m-cycles for $m \leq 68$. They used $x_i > 10^{17}$ and an m-dependent lower bound for K from a generalized lemma of Crandall [2]. The exterior lower bound for x_i plays a crucial role in both proofs. In these proofs, as in other research on the $3x+1$ problem [3], [5] and also in this article, the quantities $\Delta = 2^{K+L} - 3^K$, $\Lambda = (K+L)\log 2 - K \log 3$ (and a theoretical lower bound for these quantities) and $\delta = \log_2 3$ play an important role.

We will give a simpler (inductive) proof for the non-existence of 2-cycles. Assume that a non-trivial 2-cycle exists. Starting with $a_0 2^{k_0} - 1$ which is an odd number if $a_0 \not\equiv 0 \pmod 2$, it is easily verified that after an increasing subsequence of k_0 odd numbers, the even number $a_0 3^{k_0} - 1$ occurs. Then $a_0 3^{k_0} - 1$ is the beginning of a decreasing subsequence of (say l_0) even numbers, hence $\frac{a_0 3^{k_0} - 1}{2^{l_0}} = a_1 2^{k_1} - 1$. Simons [7] proved that a necessary and sufficient condition for the existence of a 2-cycle is the existence of a solution (a_i, k_i, l_i) of the diophantine system of equations

$$\left\{ \begin{array}{rcrcl} -3^{k_0} a_0 & + & 2^{k_1+l_0} a_1 & = & 2^{l_0} - 1 \\ 2^{k_0+l_1} a_0 & - & 3^{k_1} a_1 & = & 2^{l_1} - 1 \end{array} \right. . \tag{1}$$

Because $k_1 = K - k_0$ and $l_1 = L - l_0$, a_i (and $x_i = a_i 2^{k_i} - 1$) are lower and upper bounded functions of the variables k_0 with $1 \leq k_0 \leq K-1$ and l_0 with $1 \leq l_0 \leq L-1$. The rest of this paper is organized as follows:

- We first derive general restrictions on K and L. We use a theorem of de Weger [11] on the number of solutions of the equation $0 < 2^{K+L} - 3^K < 3^{(1-\alpha)K}$ to show that $3^K > 2^{K+L} - 3^K > 3^{0.9K} > 0$.
- We derive a numerical lower bound of $a_0 \cdot a_1 \geq 5$.
- We derive an expression for a_i as a function of the two unknown variables k_0 and l_0. We show that the product $a_0 \cdot a_1$ can be upper bounded by a function of K and L only.
- We then show that substitution of a result of de Weger's theorem into the expression for a_i leads to the upper bound $a_0 \cdot a_1 < 4$, which contradicts the lower bound $a_0 \cdot a_1 \geq 5$.
- We conclude by discussing the possibility to generalize the approach to m-cycles with $m \geq 3$ and to the $3x-1$ problem.

2 General restrictions on K and L

Simons [7] assumes a non-trivial 2-cycle. He derives from the matrix system (1) an upper bound for $\frac{2^{K+L}}{3^K}$ and proves that, for the existence of a 2-cycle, K and L must satisfy the inequality

$$0 < \Lambda = (K+L)\log 2 - K \log 3 < \sum_{i=0}^{1} \frac{1}{x_i} \ . \tag{2}$$

We exclude the trivial 2-cycle, thus $x_i \geq 3$, hence $0 < \Lambda < 2 \cdot \frac{1}{3} < 0.667 < \log 2$ or

$$0 < K + L - \delta K < 1. \tag{3}$$

For each K exactly one L value exists for which this condition is satisfied. As $\delta = \log_2 3$, the last inequality can be written as

$$3^K > 2^{K+L} - 3^K > 0. \tag{4}$$

We now apply a result of de Weger [11]. He uses a result of Waldschmidt [10] to derive an upper bound for linear forms of the type $a \log 2 - b \log 3$. He proves (theorem 5.2 on page 104 and table I on page 110) that the equation $0 < 2^{K+L} - 3^K < 3^{0.9K}$ has for $K \geq 32$ no solutions. We checked (also using his results) for $2 \leq K \leq 31$ the solutions (K, L) of

$$0 < 2^{K+L} - 3^K < 3^{0.9K}. \tag{5}$$

Based on de Weger's result and our check of small values of K we distinguish two cases:

1. $0 < 2^{K+L} - 3^K < 3^{0.9K}$. The only solutions (K, L) are $(2, 2)$, $(3, 2)$, $(4, 3)$, $(5, 3)$, $(6, 4)$, $(8, 5)$, $(10, 6)$, $(15, 9)$, $(17, 10)$ and $(29, 17)$. It can easily be verified that $(2, 2)$ corresponds with the trivial 2-cycle $(1, 2, 1, 2)$ and that for all others the system (1) has non-integer solutions a_i, so these other solutions (K, L) do not represent 2-cycles.
2. $3^{0.9K} < 2^{K+L} - 3^K < 3^K$. Solutions (K, L) must satisfy $K = 7$, 9, 11, 12, 13, 14, 16, $18 - 28$, $30 \ldots$ These solutions may represent non-trivial 2-cycles.

Thus we have as necessary conditions for the existence of a non-trivial 2-cycle: $K \geq 7$ and

$$3^{0.9K} < 2^{K+L} - 3^K < 3^K. \tag{6}$$

3 A numerical lower bound for $a_0 \cdot a_1$

Because we will use a result of Steiner, we first rephrase the main line of his proof that $(1, 2)$ is the only 1-cycle [9]. He assumes a 1-cycle with k odd and l even numbers. Then the minimum is $a2^k - 1$ and the maximum is $a3^k - 1$, hence a, k and l satisfy $\frac{a3^k - 1}{2^l} = a2^k - 1$. He then shows that, for the existence of a 1-cycle with k odd and l even numbers, k and l must satisfy

$$1 < \frac{2^{k+l}}{3^k} < \frac{2^k}{2^k - 1}. \tag{7}$$

He uses elementary number theory and classic transcendence theory [1] to derive a lower and an upper bound for k. He calculates convergents to $\frac{k+l}{k}$, checks equation (7) and shows that $k = l = 1$ is the only solution of equation (7).

Without loss of generality we may assume an increasing non-trivial 2-cycle, i.e. we have $3 \leq x_0 < x_1$. Then we have

Lemma 1 *In an increasing non-trivial* 2*-cycle,* $a_0 \neq a_1$ *and* $a_0 \cdot a_1 \neq 3$.

Proof: From the first equation of the system (1) follows

$$\frac{a_0 3^{k_0}}{a_1 2^{k_1+l_0}} = 1 - \frac{2^{l_0}-1}{a_1 2^{k_1+l_0}} > \frac{a_1 2^{k_1+l_0} - 2^{l_0}}{a_1 2^{k_1+l_0}} = \frac{a_1 2^{k_1}-1}{a_1 2^{k_1}} > \frac{a_0 2^{k_0}-1}{a_0 2^{k_0}} \geq \frac{2^{k_0}-1}{2^{k_0}}. \tag{8}$$

1. Suppose $a_0 = a_1$. Then

$$1 < \frac{a_1 2^{k_1+l_0}}{a_0 3^{k_0}} = \frac{2^{k_1+l_0}}{3^{k_0}} < \frac{2^{k_0}}{2^{k_0}-1}. \tag{9}$$

   Because (following Steiner's proof) the only solution of equation (7) is $k = l = 1$, the only solution of equation (9) is $k_0 = k_1 = l_0 = 1$. From equation (3) we find $l_1 = 1$ and from the system (1) we have $a = a_0 = a_1 = 1$. Hence $x_0 = 1$ and $x_1 = 1$ which contradicts $3 \leq x_0 < x_1$. As a result $a_0 \neq a_1$.

2. Suppose $a_0 = 1$, $a_1 = 3$. Then

$$1 < \frac{2^{k_1+l_0}}{3^{k_0-1}} < \frac{2^{k_0}}{2^{k_0}-1} < \frac{2^{k_0-1}}{2^{k_0-1}-1}. \tag{10}$$

   Because (following Steiner's proof) the only solution of equation (7) is $k = l = 1$, equation (10) has no solution.

3. Suppose $a_0 = 3$, $a_1 = 1$. Then

$$1 < \frac{2^{k_1+l_0}}{3^{k_0+1}} < \frac{2^{k_0}}{2^{k_0}-1}. \tag{11}$$

   From equation (6) follows for $K \geq 7$

$$1 + 3^{-0.1(k_0+1)} < \frac{2^{k_1+l_0}}{3^{k_0+1}} < \frac{2^{k_0}}{2^{k_0}-1}, \tag{12}$$

   which is a contradiction for $k_0 \geq 2$. For $k_0 = 1$ we have $x_0 = 5$, which ends in the cycle $(1, 2)$, so non-trivial 2-cycles cannot exist in this case.

Qed.

Since a_i is odd, $a_0 \neq a_1$ and $a_0 \cdot a_1 \neq 3$ we have

$$a_0 \cdot a_1 \geq 5. \tag{13}$$

4 An upper bound for $a_0 \cdot a_1$

Because $k_1 = K - k_0$ and $l_1 = L - l_0$, a_0 and a_1 are functions of the independent variables k_0 and l_0. From equations (1), with determinant $\Delta = 2^{K+L} - 3^K > 0$, follows

$$\begin{cases} \Delta a_0 = 3^{K-k_0}(2^{l_0}-1) + 2^{K-k_0+l_0}(2^{L-l_0}-1) \\ \Delta a_1 = 2^{k_0+L-l_0}(2^{l_0}-1) + 3^{k_0}(2^{L-l_0}-1). \end{cases} \tag{14}$$

We can derive an upper bound for the product $\Delta a_0 2^{k_0} \cdot \Delta a_1 2^{k_1}$ as a function of the real variables k_0 and l_0.

Lemma 2 $\Delta a_0 2^{k_0} \cdot \Delta a_1 2^{k_1} \le 2^K 3^K (2^L - 1)^2$ *for* $0 \le k_0 \le K$ *and* $0 \le l_0 \le L$.

Proof: As $\delta = \log_2 3$, let $u = \frac{3^{k_0}}{2^{k_0}} = 2^{(\delta-1)k_0}$ and $v = 2^{l_0}$. Then

$$\Delta a_0 2^{k_0} = f(u,v) = (\frac{3^K}{u} - 2^K)(v-1) + 2^{K+L} - 2^K, \tag{15}$$

$$\Delta a_1 2^{k_1} = g(u,v) = (\frac{2^{K+L}}{v} - 2^K)(u-1) + 2^{K+L} - 2^K \tag{16}$$

with $1 \le u \le 2^{(\delta-1)K}$ and $1 \le v \le 2^L$. For any fixed $u > 1$, f is a linear increasing function of v and g is a decreasing function of v. Also for any fixed $v > 1$, f is a decreasing function of u and g is a linear increasing function of u. We find at the boundaries ($u = 1$ or $u = 2^{(\delta-1)K}$ or $v = 1$ or $v = 2^L$)

$$f(u,1) = f(2^{(\delta-1)K}, v) = g(1,v) = g(u,2^L) = 2^K(2^L-1), \tag{17}$$

$$f(1,2^L) = g(2^{(\delta-1)K}, 1) = 3^K(2^L-1). \tag{18}$$

For any fixed u with $1 < u < 2^{(\delta-1)K}$ we consider $f \cdot g = \Delta^2 2^K a_0 \cdot a_1$ as a function of v and find

$$f \cdot g = A \cdot v + \frac{B}{v} + C, \tag{19}$$

$$\frac{d(f \cdot g)}{dv} = A - \frac{B}{v^2} \tag{20}$$

with A, B, C constants (depending on u, K, L). In particular

$$A = (\frac{3^K}{u} - 2^K)(2^{K+L} - u), \tag{21}$$

$$B = (u-1)2^{K+L}(2^{K+L} - \frac{3^K}{u}). \tag{22}$$

For $1 < u < 2^{(\delta-1)K}$ we have $A > 0$ and $B > 0$. Thus $f \cdot g$ has for $v > 0$ a global minimum at $v = \sqrt{\frac{B}{A}} > 0$ where

$$\frac{B}{A} = \frac{(u-1)2^{K+L}(2^{K+L}u - 3^K)}{(3^K - 2^K u)(2^{K+L} - u)}. \tag{23}$$

The value of v for which $f \cdot g$ is minimal can be inside or outside the interval $1 \leq v \leq 2^L$. Independent of this value, we conclude that if $1 < u < 2^{(\delta-1)K}$ and $1 \leq v \leq 2^L$, then

$$f(u, v) \cdot g(u, v) \leq \max[f(u, 1) \cdot g(u, 1), f(u, 2^L) \cdot g(u, 2^L)]. \tag{24}$$

From the above and from the definition of f and g, we find

$$\Delta^2 2^K a_0 \cdot a_1 \leq 2^K 3^K (2^L - 1)^2. \tag{25}$$

Qed.

As a consequence we have

$$a_0 \cdot a_1 \leq \frac{3^K (2^L - 1)^2}{(2^{K+L} - 3^K)^2} < \frac{3^K 2^{2L}}{(2^{K+L} - 3^K)^2}. \tag{26}$$

5 Non-existence of 2-cycles

Substitution of equations (6) and (3) into equation (26) leads to

$$a_0 \cdot a_1 < 3^{-0.8K} 2^{2L} = 2^{2L-0.8\delta K} < 2^{-0.08K+2} < 4, \tag{27}$$

which contradicts equation (13). Hence there do not exist non-trivial 2-cycles. In [7] and [8] it has already been proved that 2-cycles do not exist. The crux of our proof is that no exterior lower bound for x_i (and consequently for K) is used. Steiner's proof that 1-cycles do not exist, is explicitly used and in this respect our proof is inductive. We use transcendence theory via Steiner's proof and the theorem of de Weger. Our approach can also be applied to 1-cycles. For lemma 1 we have the trivial observation $a \geq 1$, and for lemma 2 we have $\Delta a \leq 2^L$.

6 Limitations of the approach

1. As is shown below, there is no simple generalization to m-cycles with $m \geq 3$.

   (a) Lemma 1 has a weak generalization for m-cycles.

   Lemma 3 *If in a non-trivial m-cycle we have $x_i < x_{i+1}$, then $a_i \neq a_{i+1}$.*

   Proof: In [8] a generalization of the system (1) is derived. The i^{th} equation is

$$a_{i+1} 2^{k_{i+1}} - 1 = \frac{a_i 3^{k_i} - 1}{2^{l_i}}, \tag{28}$$

from which follows, cf equation (8),

$$1 > \frac{a_i 3^{k_i}}{a_{i+1}2^{k_{i+1}+l_i}} > \frac{2^{k_i}-1}{2^{k_i}}. \tag{29}$$

Now suppose $a_i = a_{i+1}$, then we find

$$1 < \frac{2^{k_{i+1}+l_i}}{3^{k_i}} < \frac{2^{k_i}}{2^{k_i}-1}. \tag{30}$$

We can use Steiner's proof to show that for $k_i > 1, \;\; l_i > 1$ such solutions do not exist. **Qed**.

(b) In a similar way as for 2-cycles, we can exclude the case $a_i = 3$. We conclude that if $x_i < x_{i+1}$ holds for $i = 0 \dots m-2$, then (for $m \geq 8$) holds $a_0 \cdots a_{m-1} > 2^m$. The condition $x_i < x_{i+1}$ looks arbitrary in this line of reasoning. However, Simons and de Weger [8] use this condition for $i = 0 \dots m-2$ as a worst case in their proof of the non-existence of m-cycles. If this condition does not hold, then their proof effectively applies to a larger value for m.

(c) For arbitrary m, derivation of a generalized lemma 2 is not trivial. We conjecture that a generalized inequality for $a_0 \cdots a_{m-1}$ holds:

$$a_0 \cdots a_{m-1} < \frac{3^{(m-1)K}2^{mL}}{(2^{K+L}-3^K)^m}. \tag{31}$$

The approach of de Weger can be applied for any α in the interval $(0, 1)$. If $x_i \geq \frac{3}{2}m$ then $3^K > 2^{K+L} - 3^K > 3^{(1-\alpha)K} > 0$. Substitution of this result into equation (31) leads to

$$a_0 \cdots a_{m-1} < 2^{mL-(1-\alpha m)\delta K} < 2^{[m(\delta-1)-(1-\alpha m)\delta]K+m}. \tag{32}$$

For $m \geq 3$ and α arbitrary small, the coefficient of K in the exponent becomes positive and ineffective. This is a similar argument as used in [7] to explain why that proof fails for $m \geq 3$.

2. The approach can be applied to the $3x-1$ problem, defined by $T(x) = \frac{1}{2}(3x-1)$ if x is odd, $T(x) = \frac{1}{2}x$ if x is even. Several adjustment are however required and we only sketch the main line of reasoning.

(a) The $3x-1$ problem has two 1-cycles, (1) and $(5, 7, 10)$. Because for $x_0 \leq 10$ one of these cycles appears, non-trivial 2-cycles must satisfy $x_i \geq 11$.

(b) Starting with $x_0 = a_0 2^{k_0} + 1$, it turns out that the coefficients a_i must satisfy, cf equation (1),

$$\left\{ \begin{array}{rcrcl} 3^{k_0}a_0 & - & 2^{k_1+l_0}a_1 & = & 2^{l_0}-1 \\ -2^{k_0+l_1}a_0 & + & 3^{k_1}a_1 & = & 2^{l_1}-1. \end{array} \right. \tag{33}$$

$\Delta = 2^{K+L} - 3^K < 0$ and the general restriction, cf equation (2), is

$$0 < K \log 3 - (K + L) \log 2 < \sum_{i=0}^{i=1} \frac{1}{x_i} < 0.2 \; , \tag{34}$$

or

$$0 < 3^K - 2^{K+L} < \frac{1}{4} 2^{K+L} < \frac{1}{4} 3^K . \tag{35}$$

(c) From de Weger's method it follows that for $K \geq 2$ the only solutions (K, L) of

$$0 < 3^K - 2^{K+L} < 3^{0.89K} \tag{36}$$

and of equation (34) are $(2, 1)$, $(4, 2)$, $(6, 3)$, $(7, 4)$, $(9, 5)$, $(12, 7)$, $(14, 8)$, $(19, 11)$ and $(24, 14)$. The exponent $0.89K$ in this upper bound discards the pair $(53, 31)$, which satisfies $0 < 3^K - 2^{K+L} < 3^{0.9K}$, as an exception. $(2, 1)$ refers to a trivial 1-cycle, $(4, 2)$ and $(6, 3)$ refer to a trivial 2 and 3-cycle. $(7, 4)$ does refer to a non-trivial 2-cycle. Checking the system (33) against $1 \leq k_0 \leq 6$ and $1 \leq l_0 \leq 3$ reveals that the solution $k_0 = 4$, $l_0 = 1$, $k_1 = 3$, $l_1 = 3$, $a_0 = 1$, $a_1 = 5$ corresponds with the 2-cycle $(17, 25, 37, 55, 82, 41, 61, 91, 136, 68, 34)$. As a result $(14, 8)$ refers to a trivial 4-cycle. For $(9, 5)$, $(12, 7)$, $(19, 11)$ and $(24, 14)$ the system (33) has no integer solutions. Thus we have as a necessary condition for the existence of another non-trivial 2-cycle: $K \geq 3$ and

$$\frac{1}{4} 3^K > 3^K - 2^{K+L} > 3^{0.89K} > 0. \tag{37}$$

(d) A numerical lower bound for $a_0 \cdot a_1$ comes from the requirement, cf lemma 1,

$$1 < \frac{a_0 3^{k_0}}{a_1 2^{k_1 + l_0}} < \frac{2^{k_0} + 1}{2^{k_0}} < \frac{2^{k_0}}{2^{k_0} - 1} . \tag{38}$$

In a similar way as for the $3x + 1$ problem, we can exclude the cases $a_0 = a_1$ and $a_0 \cdot a_1 = 3$ and conclude that $a_0 \cdot a_1 \geq 5$.

(e) The expression of equation (26) also applies to the $3x - 1$ problem, because $-a_0$ and $-a_1$ satisfy equation (1). From here on the original proof continues and other non-trivial 2-cycles cannot exist.

3. From the above we conjecture that the approach applies to the $3x \pm q$ problem. For $q \geq 5$, non-trivial 2-cycles in general exist because of the relatively large exception class of equations (6) and (36).

4. The plausible quest for an inductive approach from $(m - 1)$-cycles to m-cycles remains open. Equation (32) gives an upper bound for real variables k_i and l_i with $0 \leq k_i \leq K$ and $0 \leq l_i \leq L$. Because m-cycles correspond to positive integer values e.g. $1 \leq k_i \leq K - (m-1)$ and $1 \leq l_i \leq L - (m-1)$, an analysis of the product expression for $m \geq 3$ for integer k_i and l_i is required. Analysis of the rounding effects becomes for large m increasingly complex because of the $2(m - 1)$ variables k_i, l_i for $i = 0 \ldots m - 2$. This is left for future research.

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