Stabilizability by state feedback implies stabilizability by encoded state feedback

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Submitted on August 26, 2003
Revised on March 3, 2004

Abstract

Encoded state feedback is a term which refers to the situation in which the state feedback signal is sampled every \( T \) units of time and converted (encoded) into a binary representation. In this note stabilization of nonlinear systems by encoded state feedback is studied. It is shown that any nonlinear control system which can be globally asymptotically stabilized by “standard” (i.e. with no encoding) state feedback can also be globally asymptotically stabilized by encoded state feedback, provided that the number of bits used to encode the samples is not less than an explicitly determined lower bound. By means of this bound, we are able to establish a direct relationship between the size of the expected region of attraction and the data rate, under the stabilizability assumption only, a result which – to the best of our knowledge – does not have any precedent in the literature.

Keywords: Encoded feedback, Quantization, Nonlinear control systems, Stabilizability, Communication

*This work was partially supported by NSF under grant ECS-0314004.
1 Introduction

The control scenario under consideration is as follows: a nonlinear system must be asymptotically stabilized and a communication channel lies between the sensors (which measure the full state of the system) and actuators (which provide the control effort). The channel is assumed to be a noise-less delay-free medium through which one packet of \( B \) bits travels every \( T \) units of time. Devices (encoders) capable of conveniently converting (encoding) the state vector sampled every \( T \) units of time into a meaningful packet \( s \) of \( B \) bits which can travel through the communication channel must be designed. Once the packet \( s \) is received at the other end of the channel, reconstruction (decoding) of the state sample from the encoded information contained in the packet \( s \) takes place and the reconstructed sample becomes available to the controller. Clearly, because quantization effects occur, the reconstruction procedure cannot be exact and the reconstructed sample will differ from the original sample by an encoding or quantization error. Therefore, the challenge is to devise encoders and decoders (devices which perform the decoding procedure) so that the reconstructed information can be used by the controller to stabilize the system despite of the presence of the encoding error. The controller will make use of the well-known “certainty-equivalence” principle to devise the control action based on the reconstructed information provided by the decoder.

Many contributions have appeared in the literature about the problem outlined above and sometimes called control under communication constraints, finite bit-rate control, control with limited information and so on (see, to cite a few, [4], [9], [20], [21], [3], [14], [7], [16], [2], [19], [10], [8], [1], [12], [13] and references therein). Among these contributions, papers such as [12] and [13] have focused more than others on the problem for nonlinear control systems.

Loosely speaking, it has been shown in [12] and [13] that globally stabilizing a nonlinear system by encoded state feedback is always possible provided that the system can be made input-to-state stable (ISS) with respect to measurement errors and a suitable condition on the number of bits used for encoding is satisfied. It should be stressed, though, that assuming input-to-state stability with respect to measurement errors sensibly restricts the class of systems to which the methods of [12] and [13] can be applied. However, the assumption can be relaxed in several ways. In [5] and [6], for instance, it has been examined the role of integral input-to-state stability (iISS) (a property which is known to be less restrictive than input-to-state stability – see e.g. [18]) and it has been shown that there are remarkable classes of nonlinear control systems which can be made integral input-to-state stable by “standard” (i.e. with no encoding) feedback which can also be globally asymptotically stabilized by (saturated) encoded feedback while guaranteeing some additional interesting features (such as the number of bits required to encode the feedback information being equal to the dimension of the process to control and independent of the traffic level on the channel). In this note we aim to show that any nonlinear control system which can be globally asymptotically stabilized by standard state feedback can also be globally asymptotically stabilized by encoded state feedback provided that the number \( B \) of bits used to encode information is large enough

\(^1\)Clearly, the concerns with a control system in which feedback travels through communication channels regard not only the conversion of an analog quantity into a finite sequence of bits but also the uncertain nature of the medium which might introduce noise, delays, loss of packages etc. However in this paper we shall neglect such additional features of the channel.
(cf. formula (40)). This is a remarkable improvement with respect to existing results, as we simply require the existence of a standard state-feedback stabilizing law and not a feedback law which renders the closed-loop system input-to-state stable with respect to errors on the state measurement. A comparison with the design strategy of [15] is appropriate. In [15] the authors deal with minimality of data-rate for stabilizing nonlinear discrete-time systems. The state is first confined within a suitably small neighborhood of the equilibrium point by allowing a large data-rate, and then, once the state has entered the neighborhood, a lower data-rate is used. By exploiting the linearization of the system around the equilibrium point, the lower data-rate is actually shown to be minimal when the control objective is local uniform exponential stabilization. Although it may be interesting to derive analogous results for continuous-time nonlinear systems in view of the arguments presented in this contribution, this goes beyond the scope of the paper and is not investigated here. We stress, though, that by means of the bound (40) we are able to establish a direct relationship between the size of the expected region of attraction and the data rate, under the stabilizability assumption only, a result which – to the best of our knowledge – does not have any precedent in the literature.

The paper is organized as follows. Section 2 contains more details on the problem and in particular on the encoding/decoding procedure. Section 3 deals with asymptotic state estimation from encoded state feedback. The main result (Proposition 1), together with a number of technical lemmas needed for its proof, is stated in Section 4. The last section, Section 5, draws the conclusions.

2 Preliminaries

Consider a system of the form
\[ \dot{x} = f(x, u) \tag{1} \]
with \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \) and \( f(\cdot, \cdot) \) a smooth map.

We shall assume that the compact set to which the initial condition \( x(0) \) belongs is a known hyper-cube \(^2\) in \( \mathbb{R}^n \), denoted \( \mathcal{C}_0^X \), with centroid the origin \( 0 \in \mathbb{R}^n \) and edges of length \( 2X \), where \( X \) is a non-negative real number. In particular, we have \(^3\)

\[ |x(0)|_\infty \leq X. \]

We shall denote by \( W \geq X, \bar{W} := W + X, F, U \) some positive real numbers for which

\[ |f(x, u) - f(\bar{x}, u)|_\infty \leq F|x - \bar{x}|_\infty \tag{2} \]

for all \( x, \bar{x}, u \) such that

\[ |x|_\infty, |\bar{x}|_\infty \leq \bar{W}, \quad |u|_\infty \leq U. \tag{3} \]

We also set \(^{[13]}\)

\[ \Lambda := e^{FT}. \]

\(^2\)The case in which such a set is unknown is discussed at the end of Section 4.

\(^3\)For a vector \( y \in \mathbb{R}^n \) the symbol \( |y|_\infty \) denotes the \( \infty \)-norm of vector \( y \), i.e. the quantity \( \max_{1 \leq i \leq n} |y_i| \).
Having introduced the system, we proceed to introduce encoders and decoders. First of all it must be specified that the functioning of the encoder and the decoder is substantially based on the so-called quantization region $\Omega$, a subset of the state space $\mathbb{R}^n$.

Quantization region. The quantization region $\Omega$ is a hyper-cube centered around the centroid $\bar{x}$ in which each edge has length $L$ — the latter is called the range of the quantization region. Hence, $\Omega = C_{\bar{x}}L^2/2$. State $x$ belongs to $\Omega$ if and only if

$$|x - \bar{x}|_\infty \leq \frac{L}{2}.$$  (4)

Both $\bar{x}$ and $L$ are updated every $T$ units of time. The way in which this update takes place is established by the encoder and the decoder — see formulas (5), (6) below. Each edge of the quantization region is uniformly partitioned into the same number $N > 1$ of parts. Therefore the quantization region will be uniformly partitioned into $N^n$ sub-regions or smaller hyper-cubes, each one with its own centroid easily computable from the knowledge of $\bar{x}$, $L$ and $N$. The number $N$ is used to define the quantity

$$\frac{\Lambda}{N} =: R.$$  

Remark. Observe that number $R$ can be set equal to any (arbitrarily small) positive number by increasing the number of bits used to encode the state information. Indeed, this number of bits is given by $B := \lceil \log_2(N^n) \rceil$. Having $n$ — the dimension of the state space of the system — fixed, increasing $B$ means increasing $N$. $\triangleright$

Having described the structure of the quantization region, encoder/decoder equations for the centroid and the range can be given as follows. We are implicitly assuming that, given an initial condition $x(0)$ and a control input $u(\cdot)$, the update laws below are defined only on the maximal interval of existence of the equations involved.

Encoder ([13], [19]). The centroid update law is:

$$\frac{d}{dt} \bar{x}(t) = f(\bar{x}(t), u(t)) , \quad t \in [kT,(k+1)T) , \quad k \geq 0 ,$$

$$\bar{x}(kT) = \hat{x}(kT) , \quad k \geq 0 ,$$  (5)

with initial condition $\bar{x}(0^-) = 0$ and where $\hat{x}(kT)$ is the centroid of the sub-region of $\Omega(kT)$ — defined by the centroid $\bar{x}(kT^-)$ and range $L(kT)$ — where $x(kT)$ lies. (See below for more details on how $\hat{x}(kT)$ is determined.) The range update law is:

$$L((k+1)T) = RL(kT) , \quad k \geq 0 ,$$

$$L(0) = 2X.$$  (6)

Remark. The reason for adopting a (dynamic) quantizer more complex than e.g. a static one is mainly due to the fact that we aim to achieve asymptotic and not only practical stabilization under arbitrary sampling time. $\triangleright$

Remark. Equation (5) requires the control law $u(\cdot)$ to be known to the encoder, a requirement which may not be always satisfied in view of the physical separation between
sensors (encoder) and actuators. However, in the present scenario, this is not a stringent requirement, since $u(\cdot)$ is a control action which can be exactly reconstructed by the encoder starting from the knowledge of $\hat{x}(\cdot)$ and the control map (i.e. map $k(\cdot)$ in (9) — see below).

At each time $kT$, quantities $x(kT)$, $\bar{x}(kT^-)$ and $L(kT)$ are available to the encoder which uses them to construct the quantization region $\Omega(kT)$. If $x(kT) \in \Omega(kT)$, the encoder determines the sub-region of the quantization region where $x(kT)$ lies. This sub-region has a centroid $\hat{x}(kT)$ whose expression is given by

$$\hat{x}(kT) = \bar{x}(kT^-) + \begin{bmatrix} \tilde{b}_1(kT)L(kT)/2N \\ \vdots \\ \tilde{b}_n(kT)L(kT)/2N \end{bmatrix}$$

(7)

where the $\tilde{b}_i(kT)$'s are suitable integers taking values in the set

$$\{-N, \ldots, -5, -3, -1, +1, +3, +5, \ldots, +N\}$$

if $N$ is an even integer, or in the set

$$\{-N, \ldots, -6, -4, -2, 0, +2, +4, +6, \ldots, +N\}$$

if $N$ is an odd integer. In either case, vector

$$\tilde{b}(kT) := [\tilde{b}_1(kT) \tilde{b}_2(kT) \ldots \tilde{b}_n(kT)] \in \mathbb{R}^n$$

can take on $N^n$ possible values. If $x(kT) \notin \Omega(kT)$, that is $x(kT)$ lies in the overflow region $\mathbb{R}^n \setminus \Omega(kT)$, then $\tilde{b}(kT)$ must take on an additional value denoting overflow. Hence, $\tilde{b}(kT)$ can be represented by a binary number if $\lceil \log_2(N^n + 1) \rceil$ bits are used. The binary number representing $\tilde{b}(kT)$ is indeed the symbol $s(kT)$ to be sent through the channel. We do not proceed further to specify the actions taken by the encoder and the decoder in the event that an overflow occurs because, as we shall see later (see first remark after Lemma 1 and also the remark after Lemma 3), by construction overflow is guaranteed to never occur.

**Decoder.** The decoder at the other end of the channel performs an inverse operation with respect to the one performed by the encoder. If the received symbol $s(kT)$ denotes overflow, then the decoder infers that overflow is occurring. Otherwise, the decoder reconstructs the vector $\hat{x}(kT)$ from $s(kT)$. First of all, from $s(kT)$ the vector $\tilde{b}(kT)$ can be promptly derived and therefore $\hat{x}(kT)$ can be calculated by (7) once $\bar{x}(kT^-)$ becomes available. Vector $\bar{x}(kT^-)$ is indeed available to the decoder, for it implements the same update laws (5), (6) as the encoder.

**Remark.** The compact set of initial conditions of the system to control must be known to the decoder.

**Controller.** As far as the controller is concerned, since we are interested in stabilizing system (1) by encoded state feedback, it is quite natural to assume that the stabilization problem is at least solvable by standard state feedback:

5
**Assumption 1** There exists a smooth feedback law
\[ u = k(x) \] (8)
such that the closed-loop system
\[ \dot{x} = f(x, k(x)) \]
is globally asymptotically stable.

**Remark.** The assumption can be replaced by a weaker assumption of semi-global stabilizability, but this will not be pursued further here. 

The controller candidate to solve the stabilization problem by encoded state feedback is “inspired” by the principle of certainty equivalence and chosen as
\[ u = k(\bar{x}) \] (9)
where \( \bar{x} \) is the “feedback” signal generated through (5)-(6). The remaining sections of the paper will be concerned with showing that controller (9) asymptotically stabilizes system (1).

## 3 Asymptotic estimation

This section is devoted to show that it is possible to exponentially reconstruct the state of system (1) from encoded state information. The result is related to the arguments found in the proof of [13], Theorem 1. The statement below keeps separated estimation and control and points out that, under boundedness of the state and of the control input, exponential state estimation from encoded information is always possible.

**Lemma 1** Given any \( X > 0 \), suppose the solution of system (1), with initial condition \( |x(0)|_\infty \leq X \) and input \( u(\cdot) \) for which \( |u(t)|_\infty \leq U \) for all \( t \geq 0 \), satisfies
\[ |x(t)|_\infty \leq W, \quad \forall t \geq 0. \]

If \( R < 1 \), then the estimate \( \bar{x}(\cdot) \) generated by the encoder/decoder (5), (6) exists for all \( t \geq 0 \) and satisfies the inequality
\[ |x(t) - \bar{x}(t)|_\infty < e^{-\lambda t} X, \quad \forall t \geq 0, \] (10)
where
\[ \lambda = \frac{|\ln R|}{T}. \]

**Remark.** In view of (10) and the definitions of the constant \( \lambda \) and of the quantization region \( \Omega(kT) \), it is not hard to see that, under the assumptions of Lemma 1, \( x(kT) \in \Omega(kT) \) for each \( k \geq 0 \), that is overflow never occurs. \( \diamond \)
Remark. A slight modification of the proof below shows that, under the hypotheses of the lemma, if the bounds on \( x(\cdot) \) and \( u(\cdot) \) are satisfied only for \( t \in [0, \bar{t}] \), with \( \bar{t} < +\infty \), rather than for all \( t \geq 0 \), then the conclusion of the Lemma holds for \( t \in [0, \bar{t}] \). ◁

Proof. Define \( \bar{t} \) the largest time for which the estimate \( \bar{x}(\cdot) \) exists and satisfies

\[
|x(t) - \bar{x}(t)|_{\infty} \leq X , \quad \forall t \in [0, \bar{t}] , \tag{11}
\]

and note that, by definition of the quantization region \( \Omega(0) \), \( x(0) \) belongs to \( \Omega(0) \) and hence

\[
|x(0) - \bar{x}(0)|_{\infty} \leq \frac{L(0)}{2N} = \frac{X}{N} < X . \tag{12}
\]

This shows that such a time \( \bar{t} \) exists. Boundedness of \( x(\cdot) \) and (11) imply

\[
|\bar{x}(t)|_{\infty} \leq X + W = \bar{W} , \quad \forall t \in [0, \bar{t}] . \tag{13}
\]

Suppose \( \bar{t} \in (0, T) \). For \( t \in [0, \bar{t}] \), by exploiting the Gronwall-Bellman lemma,

\[
|x(t) - \bar{x}(t)|_{\infty} \leq \Lambda |x(0) - \bar{x}(0)|_{\infty} \leq \Lambda \frac{X}{N} < X . \tag{14}
\]

By definition of \( \bar{x}(T) \) (see (5)), \( \bar{x}(T) \) exists provided that \( x(T) \) lies within the quantization region \( \Omega(T) \). Thus, we proceed to study the evolution of the system at time \( T \). From (14), by letting \( t \to T^- \), we obtain

\[
|x(T^-) - \bar{x}(T^-)|_{\infty} \leq L(T)/2 ,
\]

which implies that \( x(T) \) actually lies within the quantization region \( \Omega(T) \). Therefore it is true that

\[
|x(T) - \bar{x}(T)|_{\infty} \leq \frac{L(T)}{2N} = \frac{RL(0)}{2N} = \frac{\Lambda X}{N^2} < X .
\]

As a consequence, it is also true that \( \bar{t} \neq T \). We now iterate these arguments proceeding by induction. Assume that, for some \( k \geq 1 \),

\[
|x(kT) - \bar{x}(kT)|_{\infty} \leq \frac{L(kT)}{2N} ,
\]

and

\[
\bar{t} \neq kT .
\]

Suppose \( \bar{t} \in (kT, (k+1)T) \). As before, boundedness of \( x(\cdot) \) and (11) imply (13) to hold. By the Gronwall-Bellman lemma, for \( t \in [kT, \bar{t}] \),

\[
|x(t) - \bar{x}(t)|_{\infty} \leq \Lambda |x(kT) - \bar{x}(kT)|_{\infty} \leq \Lambda \frac{R^k X}{N} < X ,
\]
thus contradicting the definition of $\bar{t}$. Therefore, for all $t \in [kT, (k+1)T)$ we have
\[
|x(t) - \bar{x}(t)|_\infty \leq \Lambda |x(kT) - \bar{x}(kT)|_\infty \leq \Lambda \frac{L(kT)}{2N} = \frac{L((k+1)T)}{2}.
\] (15)
Formula (15) as $t \to ((k+1)T)^-$ shows that $x((k+1)T)$ lies within the quantization region $\Omega((k+1)T)$ and as a consequence
\[
|x((k+1)T) - \bar{x}((k+1)T)|_\infty \leq \frac{L((k+1)T)}{2N},
\] (16)
which in turn shows
\[
\bar{t} \neq (k+1)T,
\]
since, by (16),
\[
|x((k+1)T) - \bar{x}((k+1)T)|_\infty \leq \frac{L((k+1)T)}{2N} = \frac{R^{k+1}X}{N} < X.
\] (17)
This is the end of the inductive argument which allows to draw the conclusion that for all $t \in [kT, (k+1)T)$, for all $k \geq 0$, the estimate $\bar{x}(\cdot)$ exists and satisfies
\[
|x(t) - \bar{x}(t)|_\infty \leq R^{k+1}X.
\] (18)
The thesis follows by noting that
\[
R^{k+1} = \left(e^{-\ln R}\right)^{k+1} = e^{-\ln R (k+1)T/T} = e^{-\lambda(k+1)T} < e^{-\lambda t}, \text{ for all } t \in [kT, (k+1)T). \]

4 Semi-global asymptotic stabilizability by encoded state feedback

We have anticipated in the previous sections that, when only encoded state information is available, following the certainty equivalence principle, the candidate control law is naturally chosen as
\[
 u = k(\bar{x}).
\] (19)
The corresponding closed-loop system (1), (19) can be written in the following way:
\[
\dot{x} = f(x, k(x)) + f(x, k(\bar{x})) - f(x, k(x)) = f(x, k(x)) + g(x, \bar{x})(x - \bar{x})
\] (20)
where $g(\cdot, \cdot)$ is a suitable smooth map \(^4\). The key idea to prove asymptotic stability for the closed-loop system above relies on introducing a suitable Lyapunov function and consists of: (i) Establishing a (level) set within which we would like the response of the closed-loop system to evolve; (ii) Estimating the number of bits which guarantees a sufficiently small quantization error at the time when the state of the system reaches the boundary of

\(^4\)See Appendix A.
the prescribed set. This is used to show that the derivative of the Lyapunov function is strictly negative at the boundary of the set and that the response of the closed-loop system remains confined therein. Asymptotic stability is then concluded by means of fairly standard arguments.

We now introduce the Lyapunov function. As a consequence of Assumption 1, there exist a smooth function $V(·) : \mathbb{R}^n \rightarrow \mathbb{R}^+$ and class-$\mathcal{K}_\infty$ functions $\alpha(·)$, $\bar{\alpha}(·)$ and $\alpha(·)$ for which

$$\alpha(|x|_{\infty}) \leq V(x) \leq \bar{\alpha}(|x|_{\infty})$$

Set now

$$c := \bar{\alpha}(X) ,$$

and note that the following inclusion clearly holds

$$\Gamma_c := \{x : V(x) \leq c\} \supseteq C^X_0.$$  

The number $W, \bar{W}, U$ introduced in Section 2 are now given an explicit expression as follows:

$$W := \alpha^{-1}(c + 1) , \quad \bar{W} := W + X , \quad U := \max_{x : |x|_{\infty} \leq W} |k(x)|_{\infty} + 1 .$$

The number $F$ is then chosen so that (2) and (3) hold and $\Lambda, R$ are fixed accordingly.

Following [11], before introducing the main result of the paper, namely that the control law (19) asymptotically stabilizes system (1), a number of technical results which will lead to its proof are needed. These results (Lemma 2-3 and Corollary 1) refer to the situation in which $|x(0)|_{\infty} \leq X$, $R < 1$ and $\bar{x}(·)$ in system (20) is generated by the encoder/decoder (5) (with $u(·) = k(\bar{x}(·))$), (6). Note that, under Assumption 1, the solution to system (5) with $u(·) = k(\bar{x}(·))$ exists over each interval $[kT, (k + 1)T)$ for which vector $\hat{x}(kT)$ is available. This implies that, denoted by $[0, t_{max})$, with $t_{max} \leq +\infty$, the maximal interval of existence of the closed-loop system (20), the solution of system (5) with $u(·) = k(\bar{x}(·))$ surely exists over $[0, t_{max})$ (or possibly larger intervals) as well.

The first result is as follows:

**Lemma 2** Set:

$$M := \max_{x \in C^W_0} \left| \frac{\partial V}{\partial x} g(x, y) \right|_{\infty} .$$

There exists a finite time $\theta := (2MX)^{-1}$ such that, for all $t \in [0, \theta]$, $x(t) \in \text{int}(\Gamma_{c+1})$ and:

$$\frac{\partial V}{\partial x} f(x(t), k(\bar{x}(t))) < -\alpha(|x(t)|_{\infty}) + M|x(t) - \bar{x}(t)|_{\infty} .$$

**Proof.** First of all note that $|x(0)|_{\infty} \leq X < W$ and inequality $|x(0) - \bar{x}(0)|_{\infty} \leq \frac{X}{N} < X$ (see (12)) imply $|\bar{x}(0)| \leq \bar{W}$. Hence, it is possible to consider the largest interval of time $\mathcal{I}$ over which $x(·)$ exists and the following hold:

$$|x(t)|_{\infty} \leq W , \quad |\bar{x}(t)|_{\infty} \leq \bar{W} , \quad \forall t \in \mathcal{I} .$$

(26)
Two cases are possible: $\mathcal{I} = [0, +\infty)$ or $\mathcal{I} = [0, \bar{t}]$, with $\bar{t} > 0$ a suitable finite real number. We now state a number of properties which hold over $\mathcal{I}$ (regardless of the fact that $\mathcal{I}$ is finite or an infinite interval). By the inequalities in (26),

$$|x(t) - \bar{x}(t)|_\infty < e^{-\lambda t} X.$$  \hspace{1cm} (27)

By (24) and (27), we have:

$$\frac{\partial V}{\partial x} f(x(t), k(\bar{x}(t))) = \frac{\partial V}{\partial x} f(x(t), k(x(t))) + \frac{\partial V}{\partial x} g(x(t), \bar{x}(t))(x(t) - \bar{x}(t))$$

$$\leq \frac{\partial V}{\partial x} f(x(t), k(x(t))) + M|x(t) - \bar{x}(t)|_\infty$$

$$< -\alpha(|x(t)|_\infty) + MX.$$ \hspace{1cm} (28)

By integrating (25), we obtain:

$$V(x(t)) - V(x(0)) < MXt.$$ \hspace{1cm} (29)

If $\mathcal{I}$ is infinite, then $x(t) \in \text{int}(\Gamma_{c+1/2})$ for all $t \in [0, \theta]$. If $\mathcal{I} = [0, \bar{t}]$, then (29) holds for all $t \in [0, \bar{t}]$. If $\bar{t} \leq (2MX)^{-1}$, for all $t \in [0, \bar{t}]$ we would have $x(t) \in \text{int}(\Gamma_{c+1/2})$ and as a consequence

$$|x(\bar{t})| < W.$$ \hspace{1cm} (30)

On the other hand, by (27), $|x(\bar{t}) - \bar{x}(\bar{t})|_\infty < X$. Hence, $|\bar{x}(\bar{t})|_\infty < \bar{W}$. The latter and (30) contradicts the definition of $\bar{t}$. We conclude that, in the case $\mathcal{I}$ is a finite interval, then necessarily $\bar{t} > (2MX)^{-1}$ and $x(t) \in \text{int}(\Gamma_{c+1/2})$ for all $t \in [0, \theta]$. Keeping in mind the first inequality in (28), also (25) remains proven. <

An easy consequence of the latter result is the following:

**Corollary 1** For all $t \in [0, \theta]$,

$$|x(t) - \bar{x}(t)|_\infty < e^{-\lambda t} X.$$ \hspace{1cm} (31)

Moreover, for each $\epsilon > 0$ there exists a value of $N > 0$, depending on $X, T, \epsilon$, namely

$$N \geq \Lambda(X/\epsilon)^{2MXT},$$ \hspace{1cm} (32)

such that

$$|x(\theta) - \bar{x}(\theta)|_\infty < \epsilon.$$ \hspace{1cm} (33)

Finally, for all $\tilde{\theta} > \theta$,

$$x(t) \in \text{int}(\Gamma_{c+1}) \forall t \in [\theta, \tilde{\theta}] \Rightarrow |x(t) - \bar{x}(t)|_\infty \leq \epsilon \forall t \in [\theta, \tilde{\theta}].$$ \hspace{1cm} (34)

**Proof.** Consider the interval of time $\mathcal{I}$ introduced in the proof of Lemma 2. If $\mathcal{I} = [0, +\infty)$, then Lemma 1 has already proven that (31) holds for all $t \geq 0$. On the other hand, if $\mathcal{I} = [0, \bar{t}]$, then necessarily $\bar{t} > \theta$ (see final part of the proof of Lemma 2) and the same reasoning as above, this time over the interval $[0, \theta]$, yields (31) to hold for all $t \in [0, \theta]$. In particular, at $t = \theta$ we have

$$|x(\theta) - \bar{x}(\theta)|_\infty < e^{-\lambda \theta} X.$$ \hspace{1cm} (35)
Recalling that \( \lambda = |\ln R|/T \), with \( R = \Lambda/N \), we see that there always exists a (sufficiently large) value of \( N \), depending on \( X, T, \epsilon \), such that
\[
e^{-\lambda \theta} X \leq \epsilon.
\] (36)

In particular, keeping in mind the expression of \( \theta \) found in the statement of Lemma 2, it is immediate to see that the inequality \( e^{-\lambda \theta} X \leq \epsilon \) is fulfilled for values of \( R \) which satisfy
\[
R \leq e^{-2MX T \ln(X/\epsilon)} = (X/\epsilon)^{-2MXT},
\]
where without loss of generality one can assume \( X/\epsilon > 1 \) (if not, then any value of \( R < 1 \) does the job), from which we derive the lower bound (32) for \( N \). Note that if \( I \) is a semi-infinite interval this also proves the third part of the statement since (31), (35), (36) hold (the first one for all \( t \geq 0 \)) and noting that the function on the right-hand side of (31) is a strictly decreasing function of time.

In the case \( I = [0, \bar{t}] \), the third part of the statement can be proven by considering two possible cases: \( \bar{\theta} \leq \bar{t} \) and \( \bar{\theta} > \bar{t} \). In the former case, we have immediately the thesis since (31) holds for all \( t \in [0, \bar{t}] \) (see first part of the proof of Lemma 2). In the latter case we have that \( \bar{t} \in (\theta, \bar{\theta}) \) and \( x(t) \in \text{int}(\Gamma_{c+1}) \) for all \( t \in [\theta, \bar{\theta}] \). As a consequence, \( |x(\bar{t})|_\infty < W \) and \( |x(\bar{t}) - \bar{x}(\bar{t})|_\infty < \exp(-\lambda \bar{t})X < X \). But then \( |\bar{x}(\bar{t})|_\infty < W \) which contradicts the definition of \( \bar{t} \). Hence, necessarily \( \bar{t} \geq \bar{\theta} \), a case which has been already examined. This ends the proof. \( \triangleright \)

Remark. We note that the number of bits \( B = \lceil \log_2(N^n + 1) \rceil \) must be large enough so as to guarantee the fulfillment of (32). \( \triangleright \)

Lemma 3 For each positive \( \rho < c + 1 \), if
\[
N \geq \Lambda(MX/\alpha \circ \bar{\alpha}^{-1}(\rho))^{2MXT},
\] (37)
with \( M \) defined as in (24), then \( x(t) \in \text{int}(\Gamma_{c+1}) \) and
\[
|x(t) - \bar{x}(t)|_\infty < e^{-\lambda t}X
\] (38)
for all \( t \geq 0 \).

Remark. It is worth stressing that, while in Corollary 1 the estimate (38) holds for \( t \in [0, \theta] \), in this Lemma the estimate holds for all \( t \geq 0 \). In fact, the choice (37) of the data rate causes the time derivative of the Lyapunov function to be strictly negative in the vicinity of the boundary of the level set \( \Gamma_{c+1} \) (see the proof below) and this induces the state \( x(\cdot) \) to remain confined within the level set for all the times. The estimate (38) being true for all \( t \geq 0 \) is then a consequence of Corollary 1. \( \triangleright \)

Remark. The lemma holds provided that number \( N \) satisfies condition (37). If this is the case, then the lemma guarantees \( x(t) \in \text{int}(\Gamma_{c+1}) \) and (38) to hold for all \( t \geq 0 \). This implies that state \( x(kT) \) belongs to the quantization region \( \Omega(kT) \) for all \( k \geq 0 \) (see the proof of Lemma 1). This means that overflow never occurs and therefore there is no need for \( \tilde{b}(kT) \) to
take on an additional value to denote overflow. As a consequence, the lemma (and the main result to be introduced below) still holds if the number \( B \) of bits used to encode information is taken equal to \( \lceil \log_2(N^n) \rceil \) rather than \( \lceil \log_2(N^n + 1) \rceil \). \( \triangleright \)

**Proof.** Let \( \epsilon > 0 \) be such that

\[-\alpha \circ \bar{\alpha}^{-1}(\rho) + M \epsilon < 0.\]

By (25), Corollary 1 shows that, if (37) holds, then \( \dot{V}(x(\theta)) \) < 0 provided that

\[x(\theta) \in \Gamma_{c+1}^\rho := \{ x \in \mathbb{R}^n : \rho \leq V(x) \leq c + 1 \}.\]

As \( x(t) \in \Gamma_{c+1} \) for all \( t \in [0, \theta] \), then \( x(t) \) can not leave the level set \( \Gamma_{c+1} \) without contradicting the latter conclusion. The last part of the statement is also true in view of Corollary 1. \( \triangleright \)

We are now ready to state the main result of the paper.

**Proposition 1** Consider system (1) and let Assumption 1 hold. For each \( X > 0 \) and for each \( \rho < c + 1 \), if the number of bits \( B = \lceil \log_2(N^n) \rceil \) is such that

\[N > \Lambda \text{ and } N \geq \Lambda(MX/\alpha \circ \bar{\alpha}^{-1}(\rho))^{2MXT},\]

with \( M \) defined as in (24), then the solution of the closed-loop system (20)

\[\dot{x} = f(x, k(\bar{x}))\]

from the initial condition \( |x(0)|_\infty \leq X \) and with \( \bar{x}(\cdot) \) generated by the encoder/decoder (5) (with \( u(\cdot) = k(\bar{x}(\cdot)) \)), (6) satisfies:

(i) For each \( \epsilon > 0 \), there exists \( \delta(\epsilon) > 0 \) such that \( |x(0)|_\infty \leq L(0)/2 \leq \delta(\epsilon) \) implies \( |x(t)|_\infty \leq \epsilon \) for all \( t \geq 0 \);

(ii) For each \( \epsilon > 0 \), there exists \( t(\epsilon) > 0 \) such that \( |x(t)|_\infty \leq \epsilon \) for all \( t \geq t(\epsilon) \).

**Remark.** Comparing the lower bound for \( N \) in (40) with the the lower bound for \( N \), namely \( N > \Lambda \), given in [13] in the hypothesis that the system to control can be made ISS with respect to encoding errors, it is seen that there may be a price to pay for not requiring ISS, namely a larger number of bits for encoding. In fact, if \( \alpha \circ \bar{\alpha}^{-1}(\rho)/M < X \), then our scheme requires number \( N \) to be \( (MX/\alpha \circ \bar{\alpha}^{-1}(\rho))^{2MXT} \) times as larger as the number \( N \) required in [13], provided that the number \( \Lambda \) defined in Section 2 and the number \( \Lambda \) defined in [13] coincide. On the other hand, if \( \alpha \circ \bar{\alpha}^{-1}(\rho)/M \geq X \), then our scheme requires exactly the same number of bits as in [13]. \( \triangleright \)

**Proof.** It has already been proven in the Lemmas above that, under the standing hypotheses, \( x(t) \in \text{int}(\Gamma_{c+1}) \) for all \( t \geq 0 \) and as a consequence \( |x(t)|_\infty < W, |x(t) - \bar{x}(t)|_\infty < \exp(-\lambda t)X \) (the latter in view of Corollary 1), \( |\bar{x}(t)|_\infty < \bar{W} \), \( |u(t)|_\infty < U \) (by (23)) for all \( t \geq 0 \). In particular, both \( x(\cdot) \) and \( \bar{x}(\cdot) \) are bounded and

\[g(x(t), \bar{x}(t)(x(t) - \bar{x}(t))) \to 0 \text{ as } t \to +\infty.\]
Note in particular that \( g(x(t), \bar{x}(t))(x(t) - \bar{x}(t)) \) is a piece-wise continuous function of \( t \) and that
\[
|g(x, \bar{x}(t))(x - \bar{x}(t)) - g(x', \bar{x}(t))(x' - \bar{x}(t))| \leq G|x - x'|
\]
for some \( G > 0 \), for \( x, x' \) ranging over some neighborhood of \( x = 0 \) and for all \( t \geq 0 \). The Total Stability Theorem then guarantees that for all \( \varepsilon > 0 \) there exists \( \delta_1 > 0 \) and \( \delta_2 > 0 \) such that, if \( |x(0)|_{\infty} \leq \delta_1 \) and \( |g(x, \bar{x}(t))(x - \bar{x}(t))|_{\infty} \leq \delta_2 \) for all \( |x|_{\infty} \leq \varepsilon \) and all \( t \geq 0 \), the solution of the closed-loop system (20) starting from \( x(0) \) satisfies
\[
|x(t)| \leq \varepsilon
\]
for all \( t \geq 0 \). Noting that
\[
|g(x, \bar{x}(t))(x - \bar{x}(t))|_{\infty} < \tilde{M}Xe^{-\lambda t},
\]
for all \( |x| \leq \varepsilon \) and all \( t \geq 0 \), where
\[
\tilde{M} = \max_{\substack{x \in C^{c}_{0}^{|x|} \\ y \in C^{W}_{0}}} |g(x, y)|_{\infty},
\]
and recalling that \( |x(0)|_{\infty} \leq X \) and \( L(0) = 2X \), we have that \( \delta(\varepsilon) = \min\{\delta_1, \delta_2/\tilde{M}\} \) yields Property (i).

As far as Property (ii) is concerned we observe that it is an easy consequence of the results of [17] (see also Theorem 10.3.1 in [11]). In fact, the closed-loop system (20), viewed in the form
\[
\dot{x} = f(x, k(x)) + g(x, \bar{x})(x - \bar{x}),
\]
can be interpreted as an asymptotically stable system perturbed by a term which is asymptotically converging to zero and whose response \( x(\cdot) \) from initial conditions \( x(0) \) satisfying \( |x(0)|_{\infty} \leq X \) has been proven to be bounded (in fact \( x(t) \in \text{int}(\Gamma_{c+1}) \) for all \( t \geq 0 \)). By [17] (or Theorem 10.3.1 in [11]), \( x(\cdot) \) asymptotically converges to zero. This trivially proves (ii). \( \triangleright \)

Remark. The previous result holds in the hypothesis that a bound on the \( \infty \)-norm of the initial state \( x(0) \) is known to the decoder. In this regard, the result proves semi-global stabilizability. However, for those situations in which the bound on \( x(0) \) is not available, one can think of modifying the encoder and the decoder presented in the previous sections in order to expand at each sample time the quantization region \( \Omega \) used by the encoder and the decoder until the state comes to lie within the quantization region. In particular, this can be carried out following the same arguments of [13] under hypothesis of forward completeness of \( \dot{x} = f(x, 0) \), and taking \( B \) equal to \( \lceil \log_2(N^n + 1) \rceil \). As a consequence, a bound \( X \) is determined and a time \( \bar{t} > 0 \) exists for which
\[
|x(\bar{t})|_{\infty} \leq X.
\]
At time \( \bar{t} \) both the encoder and the decoder switch back to the functioning previously described. As a result, a global version of Proposition 1 holds. \( \triangleright \)
5 Conclusion

It has been proven that any nonlinear control system which can be globally asymptotically stabilized by standard state feedback can also be globally asymptotically stabilized by encoded state feedback provided that the number of bits used to encode the information is large enough. The lower bound on the number of bits is given in (40). As expected, when compared with the analogous bound given in [13] for global stabilization by encoded state feedback of nonlinear systems which are ISS with respect to encoding errors, it is seen that the bound in (40) may be larger. The arguments found in this paper can be adapted to deal with nonlinear estimation, fault detection, stabilization and output regulation by encoded output feedback.

References


A Appendix

Set $u = k(x)$, $\bar{u} = k(\bar{x})$ and

$$\bar{f}(x, \alpha) := f(x, \alpha \bar{u} + (1 - \alpha)u).$$

We have

$$f(x, \bar{u}) - f(x, u) = \bar{f}(x, 1) - \bar{f}(x, 0) = \int_0^1 \frac{\partial \bar{f}(x, \alpha)}{\partial \alpha} d\alpha = \int_0^1 \left[ \frac{\partial f(x, y)}{\partial y} \right]_{\alpha \bar{u} + (1 - \alpha)u} (\bar{u} - u) d\alpha.$$

On the other hand, setting

$$\tilde{k}(x, \alpha) := k(\alpha \bar{x} + (1 - \alpha)x),$$
we have

\[ \bar{u} - u = \bar{k}(x, 1) - \bar{k}(x, 0) = \int_0^1 \frac{\partial \bar{k}(x, \alpha)}{\partial \alpha} d\alpha = \int_0^1 \left[ \frac{\partial k(y)}{\partial y} \right]_{\alpha \bar{x} + (1-\alpha)x} (\bar{x} - x) d\alpha . \]

Finally set

\[ g(x, \bar{x}) = -\int_0^1 \left[ \frac{\partial f(x, y)}{\partial y} \right]_{\alpha \bar{x} + (1-\alpha)u} d\alpha \cdot \int_0^1 \left[ \frac{\partial k(y)}{\partial y} \right]_{\alpha \bar{x} + (1-\alpha)x} d\alpha . \]