Tracking and Disturbance Rejection for Passive Nonlinear Systems

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Statement of Originality

I hereby certify that the work presented in this thesis is my own work and, where appropriate, contributions from other people have been acknowledged.

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Abstract

The internal model principle is applied for time-invariant passive nonlinear systems that are zero-state detectable. We solve an input disturbance rejection problem (with reference signal zero), where the disturbance can be decomposed into a finite superposition of sine waves of arbitrary but known frequencies and an L^2 signal. The proposed LTI controller assures that the state trajectories of the plant converge to zero.

We derive and use a technical result concerning the behavior of a strictly output passive system with an L^2 input, which is of independent interest. Under some mild conditions on the storage function H and under some detectability assumption, we show that the state x of a strictly output passive system converges to zero for any L^2 input.

A class of passive systems called the constant incremental passive systems is introduced. Using the result on the input disturbance rejection problem, we solve the tracking and disturbance rejection problem with constant reference signal for constant incremental passive systems.

For fully actuated passive mechanical systems, we solve the tracking and disturbance rejection problem with the same disturbance signal as above and with a twice continuously differentiable reference signal r. We assume that the reference signal r and its first two derivatives are available to the controller. We combine the internal model principle with the ideas behind the *Slotine-Li* adaptive controller. The internal model-based adaptive controller that we propose causes the state trajectories to be bounded and the tracking error to converge to zero, without any prior knowledge of the plant parameters.

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to my wife and to my family

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Chapter 1

Introduction

The tracking and/or disturbance rejection problem is a central topic in control theory, and control systems that are designed to solve this problem for certain reference and disturbance signals are usually referred to as servo systems. In a seminal work from 1975, Francis and Wonham [12] introduced the internal model principle. It was a breakthrough in the study of LTI servo systems, giving necessary and sufficient conditions on the controller to assure asymptotic tracking when the reference and disturbance signals are generated by a finite-dimensional exosystem.

The internal model principle for LTI systems suggests that a copy of the exosystem must be included in the controller. For example, to eliminate the steady-state error for step reference or disturbance signals, we need integrators in the loop. If an internal model with transfer function $1/(s^2+\omega^2)$ (with suitable multiplicity) is in the feedback loop and the closed-loop system is stable, then we obtain tracking and/or disturbance rejection for sinusoidal reference and disturbance signals of frequency ω, see for example Davison and Goldenberg [9]. If the reference and disturbance signals are periodic, then the internal model principle leads to repetitive control (see for example Hara *et al* [16], Weiss and Häfele [61], Yamamoto [63]).

The idea of internal model has been generalized for the output regulation of time-invariant nonlinear systems by Isidori *et al* [5], [17]. Assuming the solvability of the Francis-Byrnes-Isidori regulator equations and if the initial state of the plant is in the output zeroing submanifold (see [5] for the definitions), the required control input which forces the state trajectories of the plant to remain in the output zeroing submanifold can be generated by an autonomous dynamical system which is referred to as an internal model [5], if its initial state is correctly chosen. In addition to the internal model, the controller must also include a stabilizing compensator which drives the state of the closed-loop system towards its output zeroing submanifold.

Recent results on the output regulation of nonlinear systems can be found, for example, in Byrnes *et al* [6], Ding [11], Isidori *et al* [18], Delli Priscoli [10] and Serrani *et* *al* [49]. Byrnes *et al* [6] gives a general framework for the nonlinear output regulation problem using the concept of limit set. Serrani *et al* [49] solves the output regulation problem semiglobally using an adaptive internal model for a class of nonlinear systems. Adaptive internal models are used also in [11] and in [18].

Passive systems have a $\mathcal{C}^1$ storage function H (defined on the state space and is bounded from below) which has the intuitive meaning of stored energy. The input signal u and the output signal y take values in the same inner product space. We denote the state of the system at time t by $x(t)$. The defining property of a passive system is that

$$\dot{H} \leq \langle y, u \rangle, \quad \text{where } \dot{H} = \frac{\partial H(x)}{\partial x}\, \dot{x}. \tag{1.1}$$

The function H is often used as a Lyapunov function for analyzing the system stability. Many physical systems (electrical circuits, mechanical systems, etc.) are passive if the input and output variables are chosen carefully such that their product represents the flow of power into the system.

It is known that passive systems have inherent stability properties. The Lyapunov stability of the equilibrium points corresponding to $u = 0$ can be shown by using H as a Lyapunov function (see, for example, Willems [62]). A stability property that some passive systems have is L^2*-stability*, i.e., if the input u is in L^2 (for $t \geq 0$), then (for any initial state) the equations of the system have a unique solution (for all $t \geq 0$) and the output y is also in L^2 (see van der Schaft [58] for details).

It is shown in [58] that *a strictly output passive system*, i.e., a passive system where the storage function H satisfies

$$\dot{H} \leq \langle y, u \rangle - k\|y\|^2, \qquad k > 0, \tag{1.2}$$

has an L^2 gain less than or equal to $\frac{1}{k}$. Such a system is locally asymptotically stable at 0 if it is zero-state detectable [58] (the definition of asymptotic stability is presented in Chapter 2). Moreover, if $\lim_{\|x\| \to \infty} H(x) = \infty$ (i.e., H is proper) then the system is globally asymptotically stable at 0.

Many references study conditions under which a nonlinear system is passive, and when a nonlinear system can be made passive by state feedback. For affine nonlinear systems, Moylan [37] describes necessary and sufficient conditions for the system to be passive. The conditions are analogous to the Kalman-Yakubovich-Popov conditions for linear time-invariant systems. In Byrnes *et al* [7], it is shown that if a nonlinear system has relative degree one and it is minimum phase, then the system can be rendered passive by state feedback.

For passive nonlinear systems, various passivity-based control techniques have been proposed to achieve setpoint regulation, see for example Ortega *et al* [39], [40],

[38], van der Schaft [58]. The book [39] describes various methods to achieve set-point regulation for systems described by Euler-Lagrange equations, for example, by PD controllers. Lozano *et al* [33] describes control applications of dissipative systems theory. In [40] and [38], a technique called Interconnection and Damping Assignment (IDA) is introduced, where the procedure is to use state feedback such that the closed-loop system is a port-controlled Hamiltonian system with desirable properties. The book [58] presents passivity-based control for port-controlled Hamiltonian systems. Using passivity-based control for tracking reference signals remains an open problem for general passive systems, as noted in Gentili and van der Schaft [13] and in Ortega [38]. However, for fully actuated mechanical systems, a passivity-based tracking controller has been proposed by Slotine and Li [52] (see also [39] for the passivity property of the closed-loop system using the *Slotine-Li* controller).

1.1 Contributions of the Thesis

We mention below the contributions of the thesis which have been submitted as articles in journals or presented in conferences.

- Under some mild conditions on the storage function and the system equations, we show the convergence of the state of a strictly output passive nonlinear system with an L^2 input. It is described in [24, 25]. We give sufficient conditions for the completeness of time-invariant and time-varying passive nonlinear system with any L^2_{loc} input. We also propose a stability concept called L^2 system-stability, a variant to the L^2-stability concept.

- Using a simple LTI passive controller, we give a necessary and sufficient condition for the controller to solve input disturbance rejection problem where the disturbance is generated by a linear exosystem. The proposed controller assures that the state trajectories of the plant converge to zero. It is also shown that when an L^2 signal is added to the disturbance generated by the exosystem (under mild assumptions on the plant) we can still achieve disturbance rejection using the same LTI controller. These results are presented in [23, 25].

- We characterise a class of passive nonlinear systems which is the class of constant incremental passive systems. We solve the problem of tracking a constant reference signal and input disturbance rejection problem by using an LTI controller. These results are described in [19, 21].

- We identify nonlinear systems which belong to the class of constant incremental passive systems. The result is reported in [21, 22].

- Combining an LTI controller with a *Slotine-Li* type adaptive controller, we solve the problem of tracking a $\mathcal{C}^2$ reference signal and rejecting input disturbance signals generated by an exosystem plus an L^2 signal for a fully actuated mechanical system. It is presented in [23, 26].

1.2 Structure of the Thesis

Throughout the thesis, we assume that the reader is familiar with the concept of passive systems and stability notions in nonlinear systems.

In the first two sections of Chapter 2 we recall some results on the existence of a unique solution to the initial value problem of an ordinary differential equation and on invariant set theory. The last section of Chapter 2 describes the system equations and the assumptions used in the rest of the thesis.

Under some technical assumptions, we show the existence of a global unique solution of the state trajectory of the system given any L^2 signal in Chapter 3. Moreover, using the invariant set theory, we establish the convergence of the state trajectories to zero.

In Chapter 4, we deal with input disturbance rejection problem where the disturbance signal can be decomposed into a signal generated by an LTI exosystem and an L^2 signal. It is shown that the disturbance rejection problem can be solved by using an LTI controller.

In Chapter 5, we identify a class of passive systems called constant incremental passive systems. Using the properties of this class of systems, we solve the tracking problem for constant reference signals and the input disturbance rejection problem by using an LTI controller.

For fully actuated mechanical systems, it is shown in Chapter 6 that the tracking of a $\mathcal{C}^2$ reference signal and the rejection of disturbance signals can be achieved by combining a *Slotine-Li* type adaptive controller and the LTI controller as used in Chapter 4.

Chapter 2

Preliminaries

In this chapter we collect results that are known, but are not stated in the literature in the form or in the detail or in the generality in which we need them.

2.1 Notation

Throughout this thesis, the inner product on any Hilbert space is denoted by $\langle\cdot,\cdot\rangle$ and $\mathbb{R}_+ = [0,\infty)$. We refer to Khalil [29] and to van der Schaft [58] for basic concepts on nonlinear systems and on passivity theory. For a finite-dimensional vector x, we use the norm $\|x\| = \left(\sum_j |x_j|^2\right)^{\frac{1}{2}}$ and for matrices, we use the operator norm induced by $\|\cdot\|$ (the largest singular value). We denote by $\mathbf{B}_\varepsilon = \{x \in \mathbb{R}^n \,|\, \|x\| < \varepsilon\}$ the open ball with radius ε and 0 as its center. The closure of $\mathbf{B}_\varepsilon$ is denoted by $\bar{\mathbf{B}}_\varepsilon$ and the boundary of $\mathbf{B}_\varepsilon$ is denoted by $\partial\mathbf{B}_\varepsilon$. For a square matrix A, $\sigma(A)$ denotes the set of its eigenvalues.

For any finite-dimensional vector space $\mathcal{V}$ endowed with a norm $\|\cdot\|_{\mathcal{V}}$, the space $L^2(\mathbb{R}_+,\mathcal{V})$ consists of all the measurable functions $f:\mathbb{R}_+ \to \mathcal{V}$ such that $\int_0^\infty \|f(t)\|_{\mathcal{V}}^2 \mathrm{d}t < \infty$. The square-root of the last integral is denoted by $\|f\|_{L^2}$. For $f:\mathbb{R}_+ \to \mathcal{V}$ and $T > 0$, we denote by f_T the truncation of f to $[0,T]$. The space $L^2_{loc}(\mathbb{R}_+,\mathcal{V})$ consists of all measurable functions $f:\mathbb{R}_+ \to \mathcal{V}$ such that $f_T \in L^2(\mathbb{R}_+,\mathcal{V})$, for all $T > 0$. The space $\mathcal{H}^1(\mathbb{R}_+,\mathcal{V})$ consists of all functions $f \in L^2(\mathbb{R}_+,\mathcal{V})$ such that $\frac{\mathrm{d}f}{\mathrm{d}t} \in L^2(\mathbb{R}_+,\mathcal{V})$ (where $\frac{\mathrm{d}f}{\mathrm{d}t}$ is understood in the sense of distributions).

The space $\mathcal{C}(\mathbb{R}^l,\mathbb{R}^p)$ (respectively $\mathcal{C}^1(\mathbb{R}^l,\mathbb{R}^p)$) consists of all the continuous (respectively continuously differentiable) functions $f:\mathbb{R}^l \to \mathbb{R}^p$.

2.2 Existence of unique solution

We recall a result on the existence and uniqueness of the solution of the differential equation $\dot{x} = f(t,x,u)$ (see also Sontag [54, Appendix C] for details).

Definition 2.2.1 Let $f \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$. *A solution of* $\dot{x} = f(t,x,u)$ with a measurable input u on an interval $\mathcal{I}$ containing 0 is an absolutely continuous function $x : \mathcal{I} \to \mathbb{R}^n$ such that

$$x(t) - x(0) = \int_0^t f(\tau, x(\tau), u(\tau))\mathrm{d}\tau \qquad \forall t \in \mathcal{I}.$$

Theorem 2.2.2 [54] *Assume that* $u : \mathbb{R}_+ \to \mathbb{R}^m$ *is measurable,* $f \in \mathcal{C}(\mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ *and the following two conditions hold for every* $a \in \mathbb{R}^n$*:*

(S1) *There exists a constant* $c > 0$ *and a locally integrable function* $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\|f(t,x,u(t)) - f(t,y,u(t))\| \leq \alpha(t)\|x - y\|$$

for almost every $t \in \mathbb{R}_+$ *and for all* $x, y \in a + \bar{\mathbf{B}}_c$.

(S2) *There exists a locally integrable function* $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ *such that*

$$\|f(t,a,u(t))\| \leq \beta(t)$$

for almost every $t \in \mathbb{R}_+$.

Then for every $x_0 \in \mathbb{R}^n$ *there exists* $\delta > 0$ *and a unique solution of* $\dot{x} = f(t,x,u)$ *with input* u *on* $[0,\delta)$ *satisfying* $x(0) = x_0$.

This theorem is a consequence of Theorem 36 in [54]. We need this result and the corollary below in later chapters when dealing with an L^2 disturbance signal.

Corollary 2.2.3 *Suppose that* u *and* f *are as in Theorem 2.2.2 and for some* $x_0 \in \mathbb{R}^n$ *and* $0 < \delta < \infty$, $[0,\delta)$ *is the maximal interval of existence of the solution of*

$$\dot{x} = f(t,x,u), \qquad x(0) = x_0.$$

Then for every compact set $\mathcal{B} \subset \mathbb{R}^n$*, there exists* $T \in [0,\delta)$ *such that* $x(T) \notin \mathcal{B}$.

PROOF. The properties **(S1)** and **(S2)** in Theorem 2.2.2 imply that for any compact set $\mathcal{B} \subset \mathbb{R}^n$, there is a locally integrable function γ such that

$$\|f(t,x,u(t))\| \leq \gamma(t), \tag{2.1}$$

for almost every $t \in \mathbb{R}_+$ and for all $x \in \mathcal{B}$. Indeed, given any $a \in \mathcal{B}$, there exists $c > 0$ and functions α and β as in Theorem 2.2.2. Then

$$\|f(t,x,u(t))\| \leq \|f(t,a,u(t))\| + \|f(t,x,u(t)) - f(t,a,u(t))\| \leq \beta(t) + c\alpha(t),$$

for all $x \in a + \bar{\mathbf{B}}_c$ and for almost every $t \in \mathbb{R}_+$. Denote

$$\gamma_a(t) = \beta(t) + c\alpha(t),$$

which is locally integrable. Consider an open covering of $\mathcal{B}$ by sets of the form $\bar{\mathbf{B}}_{c_j} + a_j$, $a_j \in \mathcal{B}$, $j \in \{1, 2, \ldots\}$. By compactness, we can extract a finite subcovering corresponding to $j \in \{1, 2, \ldots N\}$. Choose $\gamma(t) = \max_{1 \le j \le N}\{\gamma_{a_j}(t)\}$, then γ satisfies (2.1) and it is locally integrable, since γ_j is locally integrable for each j.

We prove the corollary by contradiction. Suppose that there exists a compact set $\mathcal{B} \subset \mathbb{R}^n$ such that $x(t) \in \mathcal{B}$ for all $t \in [0, \delta)$. First, we show that $\lim_{t \to \delta} x(t)$ exists. For the compact set $\mathcal{B}$, we know that there exists a locally integrable function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that (2.1) holds. Then we have

$$\|x(t_k) - x(t_j)\| \le \int_{t_j}^{t_k} \|f(\tau, x(\tau), u(\tau))\| \,\mathrm{d}\tau \le \int_{t_j}^{t_k} \gamma(\tau)\,\mathrm{d}\tau,$$

where $t_k, t_j \in [0, \delta)$, so that $\|x(t_k) - x(t_j)\| \to 0$ as $t_k, t_j \to \delta$. Since $\mathcal{B}$ is a complete metric space, $\lim_{t \to \delta} x(t)$ exists and $x(\delta) \in \mathcal{B}$. However, we could use again Theorem 2.2.2 with δ as the initial time and $x(\delta)$ as the initial state to show the existence of a solution of (2.3) on an interval $[\delta, \eta)$, $\eta > \delta$. This shows that $[0, \delta)$ is not the maximal interval of existence of the solution of (2.3). $\square$

If $\delta < \infty$ is as in Corollary 2.2.3, then it is called *the finite escape time* of the state trajectory x starting from x_0.

2.3 Invariant set of a semiflow

Let $\mathcal{X}$ be a metric space with distance μ. A set $G \subset \mathcal{X}$ is *relatively compact* if the closure of G is compact. Let $z : \mathbb{R}_+ \to \mathcal{X}$. A point $\xi \in \mathcal{X}$ is said to be an *ω-limit point of z* if there exists a sequence (t_n) in $\mathbb{R}_+$ such that $t_n \to \infty$ and $z(t_n) \to \xi$. The set of all the ω-limit points of z is denoted by $\Omega(z)$.

A map $\pi : \mathbb{R}_+ \times \mathcal{X} \to \mathcal{X}$ is said to be a *semiflow on* $\mathcal{X}$ if π is continuous, $\pi(0, x_0) = x_0$ for all $x_0 \in \mathcal{X}$ and

$$\pi(\tau + t, x_0) = \pi(\tau, \pi(t, x_0)) \qquad \forall \tau, t \in \mathbb{R}_+,\ x_0 \in \mathcal{X}.$$

A non-empty set $G \subset \mathcal{X}$ is *π-invariant* if $\pi(t, G) = G$ for all $t \in \mathbb{R}_+$.

Proposition 2.3.1 *Let $\pi : \mathbb{R}_+ \times \mathcal{X} \to \mathcal{X}$ be a semiflow on a metric space $\mathcal{X}$. Let $x_0 \in \mathcal{X}$ and denote $z(t) = \pi(t, x_0)$. If $z(\mathbb{R}_+)$ is relatively compact, then $\Omega(z)$ is non-empty, compact, π-invariant, connected and*

$$\lim_{t \to \infty} \mu(z(t), \Omega(z)) = 0. \tag{2.2}$$

The proof is a straightforward extension from the case of finite-dimensional systems where $\mathcal{X} \subset \mathbb{R}^n$ (see, for example, La Salle [31] or Logemann and Ryan [32]). Several extension of the La Salle invariance principle to the infinite-dimensional systems can also be found in Hale [15] and Slemrod [51].

PROOF. Since $z(\mathbb{R}_+)$ is relatively compact, $\Omega(z)$ is non-empty and compact. Indeed, by relative compactness of $z(\mathbb{R}_+)$, for any sequences (t_n) in $\mathbb{R}_+$, the limit $\lim_{n\to\infty} z(t_n)$ exists and is in $\mathcal{X}$. Hence $\Omega(z)$ is non-empty. We prove the compactness of $\Omega(z)$ by showing that any sequences (ξ_n) in $\Omega(z)$ has a limit $\xi \in \Omega(z)$. Let (ξ_n) be a sequence in $\Omega(z)$. Since $\Omega(z)$ is non-empty, there exists a sequence $(t_{n,m})$ in $\mathbb{R}_+$ such that $\lim_{m\to\infty} z(t_{n,m}) = \xi_n \in \Omega(z)$ for all $n \in \mathbb{Z}$. For every $n \in \mathbb{Z}$, choose $\tau_n = t_{n,n}$ so that (τ_n) is a sequence in $\mathbb{R}_+$ and $\lim_{n\to\infty} z(\tau_n) = \xi \in \Omega(z)$. Hence

$$\lim_{n\to\infty} \xi_n = \lim_{n,m\to\infty} z(t_{n,m}) = \lim_{n\to\infty} z(\tau_n) = \xi \in \Omega(z).$$

To prove π-invariance, take $\xi \in \Omega(z)$, so that there exists a sequence (t_n) in $\mathbb{R}_+$ such that $t_n \to \infty$ and $z(t_n) \to \xi$. Take $t > 0$, then

$$\pi(t,\xi) = \lim_{n\to\infty} \pi(t, z(t_n)) = \lim_{n\to\infty} \pi(t + t_n, x_0) \in \Omega(z),$$

so that $\pi(t, \Omega(z)) \subset \Omega(z)$. To prove the opposite inclusion, take $\eta \in \Omega(z)$, so that $\eta = \lim_{n\to\infty} z(\tau_n)$ for some sequence (τ_n) with $\tau_n \to \infty$. The sequence $\pi(\tau_n - t, x_0)$ (defined for n large enough, so that $\tau_n - t > 0$) being contained in a compact set, has a convergent subsequence $\pi(\theta_n, x_0)$, where (θ_n) is a subsequence of $(\tau_n - t)$. If we put $\xi = \lim_{n\to\infty} \pi(\theta_n, x_0)$, then $\pi(t,\xi) = \eta$.

To prove (2.2), assume that (2.2) is false. Then there exists a sequence (t_n) in $\mathbb{R}_+$ such that $t_n \to \infty$ and $\mu(z(t_n), \Omega(z)) \geq \varepsilon > 0$ for all n. This is a contradiction since for a subsequence (θ_n) of (t_n), we have $z(\theta_n) \to \xi \in \Omega(z)$.

To prove that $\Omega(z)$ is connected, we use contradiction. Assume that $\Omega(z) = \Omega_1(z) \cup \Omega_2(z)$ where $\Omega_1(z)$ and $\Omega_2(z)$ are disjoint. Since $\Omega(z)$ is compact then $\Omega_1(z)$ and $\Omega_2(z)$ are compact. By compactness of $\Omega_1(z)$ and $\Omega_2(z)$, there exist open covering sets $U_1, U_2 \subset \mathcal{X}$ such that $\Omega_1(z) \subset U_1$ and $\Omega_2(z) \subset U_2$. Since the compact set $\Omega(z)$ is approached by $\pi(t, x_0)$ as $t \to \infty$ and $\pi(t, x_0)$ is continuous, then there exists a sequence (t_n) such that $\pi(t_n, x_0)$ intersect U_1 and U_2 an infinite number of time. This implies that we can extract a subsequence (τ_k) of (t_n) such that $\lim_{k\to\infty} z(\tau_k) = \xi \notin U_1 \cup U_2$. This contradicts the fact that $\Omega(z) \in U_1 \cup U_2$. $\square$

2.4 Systems description

Consider a time-invariant nonlinear plant $\mathbf{P}$ described by

$$\dot{x} = f(x,u), \qquad y = h(x), \tag{2.3}$$

where the state x, the input u and the output y are functions of $t \geq 0$, such that $x(t) \in \mathbb{R}^n$, $u(t), y(t) \in \mathbb{R}^m, m \leq n$. We assume that $f \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ with $f(0,u) = 0 \Leftrightarrow u = 0$ and $h \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^m)$ with $h(0) = 0$. We assume that there exists a storage function $H \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_+)$ such that

$$\frac{\partial H(x)}{\partial x} f(x,u) \leq \langle h(x), u \rangle. \tag{2.4}$$

The plant $\mathbf{P}$ as in (2.3) with the storage function H satisfying (2.4) is passive, which means that it satisfies (1.1) (this is easy to verify). H is called *proper* if $H(x) \to \infty$ when $\|x\| \to \infty$ or equivalently, for any constant $c > 0$ the set $\{x \in \mathbb{R}^n \,|\, H(x) \leq c\}$ is compact.

Remark 2.4.1 For the plant $\mathbf{P}$ from (2.3), $f(x,u)$ is not assumed to be an affine function of the control input u. If it were affine (as for port-controlled Hamiltonian systems [58]), then the plant would be described by

$$\dot{x} = \tilde{f}(x) + g(x)u, \qquad y = h(x), \tag{2.5}$$

where $\tilde{f} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ with $\tilde{f}(0) = 0$, $g \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^{n \times m})$, $g(0)$ has rank m and h is as in (2.3). For such a plant, the passivity condition (2.4) becomes

$$\frac{\partial H(x)}{\partial x} \left[\tilde{f}(x) + g(x)u\right] \leq \langle h(x), u \rangle,$$

which (as shown in [58]) is equivalent to the *Hill-Moylan conditions*:

$$\begin{aligned} \frac{\partial H(x)}{\partial x} \tilde{f}(x) &\leq 0, & (2.6)\\ \frac{\partial H(x)}{\partial x} g(x) &= h^T(x). & (2.7) \end{aligned}$$

$\square$

Remark 2.4.2 Figure 2.1 shows an example of a passive nonlinear plant $\mathbf{P}$ for which $f(x,u)$ is not an affine function of the input u. The plant $\mathbf{P}$ in Figure 2.1 can be described in state space as follows

$$\dot{x} = -\frac{\phi(x,u)}{L} + \frac{1}{L}u, \qquad y = x, \tag{2.8}$$

where the function $\phi \in \mathcal{C}^1(\mathbb{R}^2, \mathbb{R})$ describes a nonlinear element such as a transistor or an IGBT (u plays the role of basis or gate voltage). L denotes the inductance, the state $x(t) \in \mathbb{R}$ is the current through the inductor and the scalar input signal u is the voltage $u = V_d - V_c$. We assume that $L > 0$ and $x\phi(x,u) \geq 0$ for all x and u (thus, the nonlinear element does not produce energy). The plant **P** as in (2.8) is passive with storage function

$$H(x) = \frac{1}{2}Lx^2, \tag{2.9}$$

which is the stored energy. Indeed, $\dot{H}(x) = -x\phi(x,u) + \langle x, u\rangle \leq \langle y, u\rangle$. We shall come back to this example in Chapter 4. □

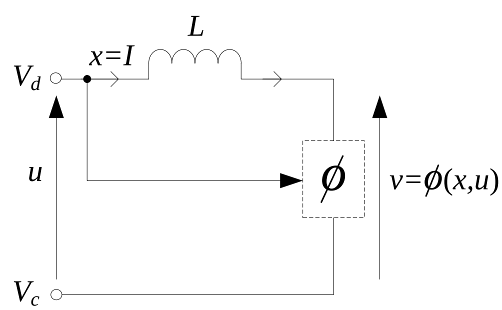


Figure 2.1: An electrical circuit that is passive and $\dot{x}$ does not depend linearly on the input u. The nonlinear component is described by $v = \phi(x,u)$.

P is said to be *zero-state detectable* if the following is true: If $u(t) = 0$ for all $t \geq 0$ and x is a solution of (2.3) such that $y(t) = 0$ for all $t \geq 0$ where x is defined, then x is defined on $[0, \infty)$ and $\lim_{t\to\infty} x(t) = 0$.

Definition 2.4.3 *Consider a system described by* $\dot{x} = f(x)$, *with* $f \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}^n)$ *and* $f(0) = 0$. *The system is*

- stable, *if for each* $\varepsilon > 0$ *there exists* $\delta(\varepsilon)$ *such that*

$$\|x(0)\| < \delta(\varepsilon) \Rightarrow \|x(t)\| < \varepsilon, \quad \forall t \geq 0.$$

- asymptotically stable, *if it is stable and there exists* $r > 0$ *such that*

$$\|x(0)\| < r \Rightarrow \lim_{t\to\infty} x(t) = 0.$$

- globally asymptotically stable, *if it is stable and* $\lim_{t\to\infty} x(t) = 0$ *for all* $x(0) \in \mathbb{R}^n$.

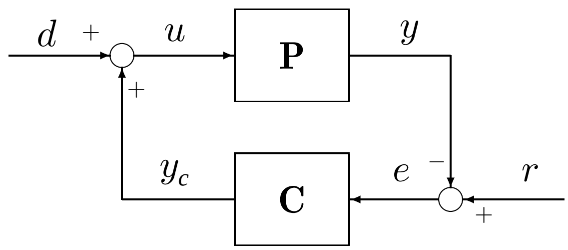


Figure 2.2: The tracking and disturbance rejection problem for the plant **P** and a certain class of signals d and r is to find a controller **C** such that the state trajectories of the closed-loop system **L** are bounded and $e(t) \to 0$ as $t \to \infty$.

The definition of the above concepts usually refers to an equilibrium point $x_0 \in \mathbb{R}^n$, but in this thesis, we consider zero to be the equilibrium point of interest.

Consider the feedback system **L** as shown in Figure 2.2, which consists of the above plant **P** with a controller **C** and with $r = 0$. If we regard y as the output function of the closed-loop system and use a proportional gain $K = K^T > kI$, $k > 0$ as our controller, i.e., $y_c = -Ky$, then the closed-loop system becomes *strictly output passive*, which means that

$$\dot{H} \leq \langle y, d \rangle - k\|y\|^2. \tag{2.10}$$

Here, $\dot{H}$ is the derivative of $H(x(t))$ with respect to t, as in (1.1).

Proposition 2.4.4 [58] *Suppose that the plant* **P** *is described by* (2.3)*, with* $H(x) > 0$ *for all* $x \neq 0$*,* $H(0) = 0$ *and* **P** *is zero-state detectable. Let the controller* **C** *be a proportional gain* $K = K^T > kI$*,* $k > 0$*. Consider the feedback system* **L** *as in Figure* 2.2*, with* $r = 0$ *and* $d = 0$*. Then* **L** *is asymptotically stable. If* H *is proper, then* **L** *is globally asymptotically stable.*

We state the proof below even though the above result is well-known in literature, for example, in [58].

Proof. The closed-loop system **L** with $r = 0$ is described by

$$\dot{x} = f(x, -Kh(x) + d), \qquad y = h(x).$$

It is easy to see that **L** is zero-state detectable. Indeed, if $d = 0$ and $y = 0$, then the closed-loop system **L** is described by $\dot{x} = f(x, 0)$. Since **P** is zero-state detectable, $\dot{x} = f(x, 0)$ and $h(x) = 0$ imply that $x(t) \to 0$ as $t \to \infty$.

It follows from (2.10) that for $r = 0$ and $d = 0$, $\dot{H} \leq -k\|y\|^2 \leq 0$ for all $t \geq 0$. This implies (using H as a Lyapunov function) that **L** is stable. It follows that there exists $\delta > 0$ such that $x(0) \in \overline{\mathbf{B}}_\delta \Rightarrow x(t) \in \overline{\mathbf{B}}_1$ for all $t \geq 0$. According to the LaSalle invariance principle (see [47], [58]), such a state trajectory x converges to the largest invariant set M contained in $\{z \in \bar{\mathbf{B}}_1 \mid \dot{H}(z) = 0\}$. In the invariant set M, H is constant along state trajectories and hence $y = 0$ along such trajectories. By the zero-state detectability of $\mathbf{L}$, all these trajectories converge to 0, hence $H(z) = H(0) = 0$ for all $z \in M$. Since $H(z) > 0$ for all $z \neq 0$, we obtain $M = \{0\}$, so that $\mathbf{L}$ is asymptotically stable.

When H is proper, then every state trajectory of $\mathbf{L}$ with $d = 0$ remains bounded, as it is easy to see. Thus, for any state trajectory x, we can apply the preceding argument with $\bar{\mathbf{B}}_1$ replaced by a ball $\bar{\mathbf{B}}_\lambda$ that contains this state trajectory. Then, we conclude that $\lim_{t\to\infty} x(t) = 0$, as claimed in the proposition. $\square$

Chapter 3

State convergence of a strictly output passive system with an L^2 input

The result presented in this chapter is based on Jayawardhana and Weiss [24, 25]. Under mild assumptions on the system's differential equations, we show that if the system is zero-state detectable and its storage function H is proper, then the state x converges to zero for any input $u \in L^2$. The result will be used in Chapter 4 to solve disturbance rejection problem where an L^2 component is added to the disturbance signal generated by the exosystem.

The convergence of the state trajectory x of $\mathbf{P}$ given a converging input signal u (i.e., $\|u(t)\| \to 0$ as $t \to \infty$) for nonlinear systems has been studied by Sontag [55]. Suppose that 0 is the global asymptotic stable (GAS) equilibrium point of $\mathbf{P}$ (taking $u = 0$). It is shown in [55] that if for a given converging input u and initial state $x(0)$, there exists a unique solution $x(t)$ of (2.3) defined for all $t \geq 0$ and x is bounded, then $x(t) \to 0$ as $t \to \infty$. This result is generalized by Ryan [44] for L^p input. Using the same assumption on the global asymptotic stability of the origin and assuming that for all compact set $K \in \mathbb{R}^n$ there exists $c > 0$ such that

$$\|f(x,u) - f(x,0)\| \leq c\|u\| \qquad \forall u \in \mathbb{R}^m, x \in K, \tag{3.1}$$

it is shown in [44] that if for a given L^p input u and initial state $x(0)$, there exists a solution of the state trajectory x defined for all $t \geq 0$ and x is bounded, then $x(t) \to 0$ as $t \to \infty$.

We extend the result of [55] and [44] in the following way: we use a technique from infinite-dimensional linear system theory to show that for any L^2 input there exists a unique solution of x for all $t \geq 0$ and we show that $x(t) \to 0$ as $t \to \infty$ using an infinite-dimensional version of the La Salle invariance principle. Here, we allow the function f to satisfy a weaker condition than (3.1) (the Lipschitz condition assumed in [44]). For example, we allow $f(x,u) = -x(1+|u|^{\frac{1}{2}})+u$ which satisfy the following condition: for each compact set $K \in \mathbb{R}^n$ there exist $c_1, c_2 > 0$ such that

$$\begin{aligned} \|f(x,u)-f(x,0)\| &= \|-x(1+|u|^{\frac{1}{2}})+u+x\| \\ &\leq c_1 + c_2\|u\| \quad \forall x \in K, u \in \mathbb{R}^m. \end{aligned}$$

The result of this chapter reveals an additional property that an L^2-stable system can have. The standard properties of an L^2-stable system are:

1. If the input is zero and the system satisfies a detectability condition, then the state converges to zero (see van der Schaft [58]).

2. Using a Bărbalat's type argument, it can be shown that if the storage function is proper then the output converges to zero for any L^2 input signal (see Teel [57]).

However, these standard stability results do not describe the convergence of the state trajectory to zero for any L^2 input.

The intuition behind the main result of this chapter is the following: According to the global asymptotic stability result in Chapter 2 for a strictly output passive system with a proper storage function H, when $u = 0$ then $x(t) \to 0$ as $t \to \infty$. If $u \in L^2$, then for very large τ the energy left in u for $t \geq \tau$ becomes negligible, and the system behaves as it would for $u = 0$, i.e., we have $x(t) \to 0$ as $t \to \infty$. However, a rigorous proof of this result is not easy. The proof of the main result uses techniques from infinite-dimensional system theory.

The technique used in this chapter can also be used for a class of nonlinear systems to show the convergence of the state given any L^p input signal, where $p \in [1, \infty)$ (see Jayawardhana [20]).

The linear version of the main result is the following: For a linear time-invariant (LTI) system $\mathbf{P}$ which is detectable and strictly output passive we have $x(t) \to 0$, for every L^2 input u. The proof of this is easy: suppose that $\mathbf{P}$ is described by

$$\dot{x} = Ax + Bu, \quad y = Cx + Du, \tag{3.2}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$. From the detectability and the strict output passivity of $\mathbf{P}$, it follows that $\mathbf{P}$ is stable, i.e., A is Hurwitz. Thus, $u \in L^2$ implies that $x \in L^2$. From (3.2), we also have that $\dot{x} \in L^2$. Using Barbălat's lemma (see Logemann and Ryan [32]), it follows that $x(t) \to 0$.

3.1 State convergence result

We consider the system **P** described by (2.3) with the mild assumptions on f and g stated after (2.3). We need additional assumptions on the function f:

(A1) For every compact set $\mathcal{B} \subset \mathbb{R}^n$, there exist constants $c_1, c_2 > 0$ such that

$$\|f(x_1,u) - f(x_2,u)\| \leq (c_1 + c_2\|u\|^2)\|x_1 - x_2\|, \tag{3.3}$$

for all $u \in \mathbb{R}^m$ and $x_1, x_2 \in \mathcal{B}$.

(A2) For every fixed $a \in \mathbb{R}^n$, there exist constants $c_3, c_4 > 0$ such that

$$\|f(a,u)\| \leq c_3 + c_4\|u\|^2 \qquad \forall u \in \mathbb{R}^m. \tag{3.4}$$

Note that **(A1)** and **(A2)** are not related to each other, i.e., one cannot be derived from the other.

Remark 3.1.1 It can be shown that **(A1)** and **(A2)** are satisfied for affine nonlinear systems **P** described by (2.5) with the assumptions stated after (2.5). $\square$

For any $\tau \geq 0$, we denote by $\mathbf{S}_\tau^*$ the left-shift operator by τ, acting on $X = L^2(\mathbb{R}_+, \mathbb{R}^m)$. The reason for this notation is that, traditionally, $\mathbf{S}_\tau$ denotes the right-shift by τ on X and $\mathbf{S}_\tau^*$ is the adjoint of $\mathbf{S}_\tau$. Suppose that $d_0 \in X$ and $d_t = \mathbf{S}_t^* d_0$, it follows that $d_t \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ for all $t \geq 0$ and the following equation holds for almost every $t \geq 0$:

$$\frac{\mathrm{d}}{\mathrm{d}t}\|d_t\|_{L^2}^2 = \frac{\mathrm{d}}{\mathrm{d}t}\int_t^\infty ||d_0(\xi)||^2 \mathrm{d}\xi = -||d_0(t)||^2. \tag{3.5}$$

Theorem 3.1.2 *Let the plant* **P** *defined by* (2.3) *be zero-state detectable and assume* **(A1)-(A2)**. *Assume that* **P** *has a storage function H such that $H(x) > 0$ for $x \neq 0$, $H(0) = 0$, H is proper and* (2.10) *(strict output passivity) holds with $k > 0$.*

Then for every initial condition $x(0) \in \mathbb{R}^n$ and for every $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, the state trajectory x of **P** *is defined uniquely for all $t \geq 0$ and it satisfies $x(t) \to 0$ as $t \to \infty$ (and hence $y(t) \to 0$ as $t \to \infty$).*

PROOF. Using **(A1)**, we have that for every compact set $\mathcal{B} \subset \mathbb{R}^n$ there exist constants $c_1, c_2 > 0$ such that (3.3) holds. By denoting $\alpha(t) = c_1 + c_2\|u(t)\|^2$ and since $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, it is easy to see that α is locally integrable and satisfies the condition **(S1)** in Theorem 2.2.2.

Using the assumption **(A2)**, we have that for each fixed $a \in \mathbb{R}^n \times \mathbb{R}^l$, there exist constants $c_3, c_4 > 0$ such that (3.4) holds. By denoting $\beta(t) = c_3 + c_4 \|u(t)\|^2$ and since $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, β is locally integrable and satisfies the condition **(S2)** in Theorem 2.2.2 for the state equation (2.3).

It follows from Theorem 2.2.2 (with α, β as above) that for any initial value $x(0) \in \mathbb{R}^n$, there exists $\delta > 0$ and a unique solution of (2.3) with input $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ on $\mathcal{I} = [0, \delta)$. In particular, x is absolutely continuous on $\mathcal{I}$.

We define an infinite-dimensional signal generator for the signal u. This signal generator has the state space $X = L^2(\mathbb{R}_+, \mathbb{R}^m)$ and the evolution of its state is governed by the operator semigroup $(\mathbf{S}^*_\tau)_{\tau \geq 0}$. Thus, the state of the signal generator at time t is $d_t = \mathbf{S}^*_t d_0$, where $d_0 \in X$ is the initial state. The generator of this semigroup is $\mathcal{A} = \frac{\mathrm{d}}{\mathrm{d}\xi}$ with domain $\mathcal{D}(\mathcal{A}) = \mathcal{H}^1(\mathbb{R}_+, \mathbb{R}^m)$. The observation operator of this signal generator is $\mathcal{C}$, defined for $\phi \in \mathcal{D}(\mathcal{A})$ by $\mathcal{C}\phi = \phi(0)$. It can be checked that $\mathcal{C}$ is admissible in the sense of Weiss [60]. We need the Lebesgue extension of $\mathcal{C}$, denoted by $\mathcal{C}_L$, defined by

$$\mathcal{C}_L \phi = \lim_{\varepsilon \to 0} \mathcal{C} \frac{1}{\varepsilon} \int_0^\varepsilon \mathbf{S}^*_t \phi \, \mathrm{d}t = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \int_0^\varepsilon \phi(\xi) \, \mathrm{d}\xi$$

with $\mathcal{D}(\mathcal{C}_L)$ being the set of all $\phi \in X$ for which the above limit exists. We refer to [60] for more information on the concept of Lebesgue extension. The output function of the signal generator is $u(t) = \mathcal{C}_L d_t$, which is defined for almost every $t \geq 0$. It turns out that $u = d_0$ (the generated signal is the initial state).

We define an extended system $\mathbf{Q}$ by connecting $\mathbf{P}$ to the generator for u as shown in Figure 3.1. Then we have

$$\begin{aligned} \dot{x}(t) &= f(x(t), u(t)), && (3.6) \\ d_t &= \mathbf{S}^*_t d_0, && (3.7) \\ u(t) &= \mathcal{C}_L d_t, && (3.8) \\ y(t) &= h(x(t)). && (3.9) \end{aligned}$$

Let $z(t) = \begin{bmatrix} x(t) \\ d_t \end{bmatrix}$ denote the state of $\mathbf{Q}$ at time t, so that $z(t) \in Z = \mathbb{R}^n \times X$.

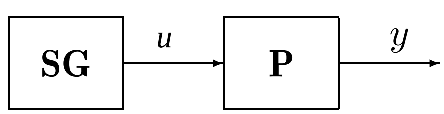


Figure 3.1: The extended closed-loop system **Q**. The block **SG** is the infinite-dimensional linear signal generator for the L^2 signal u.

Consider the storage function $H_{cl} : Z \to \mathbb{R}_+$ defined for $z = \left[\begin{smallmatrix} x \\ d \end{smallmatrix}\right]$ by $H_{cl}(z) = H(x) + \gamma \|d\|^2$ where $\gamma > \frac{1}{4k}$, where $k > 0$ is the constant from (2.4). We show that $H_{cl}(z(t))$

is absolutely continuous as a function of t. Since $H \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_+)$ and the solution x of (3.6) is absolutely continuous as a function of t defined in $\mathcal{I}$, it follows that $H(x(t))$ is absolutely continuous on $\mathcal{I}$. From (3.5) and since $d_0 \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, it follows that $\frac{\mathrm{d}}{\mathrm{d}t}\|d_t\|_{L^2}^2 \in L^1(\mathbb{R}_+, \mathbb{R}^m)$. This implies that $\|d_t\|_{L^2}^2$ is absolutely continuous on $\mathbb{R}_+$.

Using (2.10), (3.5), (3.6) – (3.8), we obtain that, for almost every $t \in \mathcal{I}$,

$$\begin{aligned} \dot{H}_{cl} &= \dot{H} - \gamma\|d_0(t)\|^2 \\ &\leq \langle y, d_0(t)\rangle - k\|y\|^2 - \gamma\|d_0(t)\|^2, \\ &\leq (\frac{1}{2\theta} - \gamma)\|d_0(t)\|^2 + (\frac{\theta}{2} - k)\|y\|^2 \quad \forall \theta > 0. \end{aligned}$$

By choosing $\theta \in (1/2\gamma, 2k)$, we obtain

$$\dot{H}_{cl}(z(t)) \leq -c_5\|u(t)\|^2 - c_6\|y(t)\|^2 \leq 0, \tag{3.10}$$

where $c_5 = \gamma - \frac{1}{2\theta} > 0$ and $c_6 = k - \frac{\theta}{2} > 0$.

Let us prove that $\mathcal{I} = \mathbb{R}_+$. If the maximal interval of definition of a state trajectory is $\mathcal{I} = [0, \delta)$ with $\delta < \infty$, then it follows from Corollary 2.2.3 that $x(t)$ must leave any compact set $\mathcal{B} \subset \mathbb{R}^n$ at some finite time $T < \delta$. Since H_{cl} is absolutely continuous as a function of t and bounded from below, (3.10) implies that $H_{cl}(z(t))$ is bounded and non-increasing for all $t \in \mathcal{I}$. In particular, the state $x(t)$ never leaves the compact set $\{x \in \mathbb{R}^n \mid H(x) \leq H_{cl}(z(0))\}$ for all $t \in \mathcal{I}$. This contradiction shows that $\mathcal{I} = \mathbb{R}_+$ and $H_{cl}(z(t))$ has a limit h as $t \to \infty$.

We will prove the relative compactness of $z(\mathbb{R}_+)$. It has been shown that $x(t)$ is bounded for all $t \in \mathbb{R}_+$, hence $x(\mathbb{R}_+)$ is relatively compact in $\mathbb{R}^n$. Since $\lim_{t\to\infty}\|d_t\|_{L^2} = 0$, the state trajectory of the signal generator $\{d_t \mid t \geq 0\}$ is relatively compact in $L^2(\mathbb{R}_+, \mathbb{R}^m)$. Therefore $z(\mathbb{R}_+)$ is relatively compact in $\mathbb{R}^n \times X$.

Let π denote the semiflow of (3.6)–(3.7) so that $z(t) = \pi(t, z_0)$. (π exists because $\mathbf{Q}$ has no state trajectories with a finite escape time.) According to Proposition 2.3.1 and the relative compactness of $z(\mathbb{R}_+)$, $\Omega(z)$ is non-empty, compact and π-invariant.

For any $\xi \in \Omega(z)$, there is a sequence (t_n) in $\mathbb{R}_+$ such that $t_n \to \infty$ and $z(t_n) \to \xi$. By the continuity of H_{cl}, $H_{cl}(\xi) = \lim_{n\to\infty} H_{cl}(z(t_n)) = h$. Therefore, $H_{cl}(z(t)) = h$ on $\Omega(z)$. Since $\Omega(z)$ is π-invariant, $\Omega(z) \subset E = \{z \mid H_{cl}(z) = h\}$.

Let M be the largest π-invariant set contained in E. Since $\Omega(z)$ is π-invariant and $\Omega(z) \subset E$, we have $\Omega(z) \subset M$.

In the invariant set M, H_{cl} is constant along state trajectories and $y = 0$ and $u = 0$ along such trajectories. By the assumptions of the theorem, $\mathbf{P}$ is zero-state detectable, i.e., if $u(t) = 0$ and $y(t) = 0$ for all $t \in \mathbb{R}_+$ then $x(t) \to 0$ as $t \to \infty$. Also, if $u(t) = 0$ for all $t \in \mathbb{R}_+$ then $d_0 = 0$, so that $d_t = 0$ for all $t \in \mathbb{R}_+$. Hence, in the invariant set M,

$H_{cl}(z) = H_{cl}(0) = 0$ for all $z \in M$, i.e., $h = 0$. Since $H_{cl}(z) > 0$ for all $z \neq 0$, we obtain that $M = \{0\}$, hence $\Omega(z) = \{0\}$. Using (2.2) it follows that $x(t) \to 0$ as $t \to \infty$.

The above argument is valid for any $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ and for any initial state $x(0) \in \mathbb{R}^n$. □

Corollary 3.1.3 *Let the plant* **P** *be as in Theorem 3.1.2. Then for every $x(0) \in \mathbb{R}^n$ there exists a unique solution of (2.3) with $u \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ in $\mathbb{R}_+$.*

PROOF. To prove the result, we use a contradiction. Suppose that there exists an input $u \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ and a finite escape time $T > 0$ for the trajectory of x of the system with initial conditions $x(0) = x_0$. According to Corollary 2.2.3, $\|x(t)\| \to \infty$ as $t \to T$. Then using $\tilde{u}$ given by

$$\tilde{u}(t) = \begin{cases} u(t) & \forall t \in [0, T], \\ 0 & \forall t \in (T, \infty), \end{cases}$$

the trajectory $\tilde{x}$ of the system with $\tilde{x}(0) = x_0$ and input $\tilde{u}$ also has the same finite escape time T. This is a contradiction. Indeed, since $\tilde{u} \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, it follows from Theorem 3.1.2 that the state trajectory $\tilde{x}$ corresponding to $\tilde{u}$ is bounded for $t \in [0, \infty)$, i.e., there is no finite escape time. □

Note that the convergence of the state trajectory x to zero does not imply that $x \in L^2(\mathbb{R}_+, \mathbb{R}^n)$. We give an example where $u \in L^2(\mathbb{R}_+, \mathbb{R}^m) \Rightarrow y \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ with a unique solution of the state $x(t)$ for all $t \in \mathbb{R}_+$, but $x \notin L^2(\mathbb{R}_+, \mathbb{R}^n)$. Let the strictly output passive plant **P** be described by

$$\dot{x} = -x^3 + u, \quad y = x^3, \tag{3.11}$$

where $x(t), u(t), y(t) \in \mathbb{R}$. Using the storage function $H(x) = \frac{1}{4}x^4$, it follows from Theorem 3.1.2 that for every $u \in L^2(\mathbb{R}_+, \mathbb{R})$ and every initial state $x(0) \in \mathbb{R}$, there exists a unique solution $x(t)$ of (3.11) in $\mathbb{R}_+$ and $\lim_{t\to\infty} |x(t)| = 0$. However, this does not imply that $x \in L^2(\mathbb{R}_+, \mathbb{R})$. Using $u = 0$ and initial state $x(0) = a$, the solution x of (3.11) is given by

$$x(t) = \left(2t + \frac{1}{a^2}\right)^{-0.5},$$

so that $x \notin L^2(\mathbb{R}_+, \mathbb{R})$.

3.2 L^2 system-stability

Consider the following single-input single-output plant **P**

$$\dot{x} = -xu^{2p} - x + u, \quad y = x, \tag{3.12}$$

where p is a positive integer. This plant $\mathbf{P}$ is strictly output passive. Indeed, using the storage function $H(x) = \frac{1}{2}x^2$, we have

$$\begin{aligned}\dot{H} &= -x^2u^{2p} - x^2 + xu \\ &\leq \langle y, u\rangle - \|y\|^2.\end{aligned}$$

From this inequality, it can be shown that for any state trajectory x which is defined on $[0,T)$, $\mathbf{P}$ has a finite L^2 gain of 1, i.e., $\|y_T\|_{L^2} \leq \|u_T\|_{L^2} + \sqrt{2H(x(0))}$ (see Lemma 6.5 in [29] for details).

However, this does not imply that for every $u \in L^2(\mathbb{R}_+,\mathbb{R})$ the solution $x(t)$ of (3.12) exists for $t \in [0,T)$, $T > 0$ (hence the output function y is not well defined). Suppose that $u \in L^2$ is given by

$$u(t) = \begin{cases} t^{-0.25} & t \in [0,1) \\ t^{-1} & t \in [1,\infty) \end{cases}, \tag{3.13}$$

so that $u \in L^2(\mathbb{R}_+,\mathbb{R})$. Now the state equation (3.12) can be written as follows:

$$\dot{x} = -xt^{-0.5p} - x + t^{-0.25} \quad \forall t \in [0,1). \tag{3.14}$$

It can be shown that if $x(0) \neq 0$ and $p = 2$ then the solution of (3.14) does not exist. Without loss of generality, assume that $x(0) < 0$. Using contradiction, suppose that there exists a solution x of (3.14) on $[0,\delta)$ with $p = 2$. By the continuity of x on $[0,\delta)$, there exists $\varepsilon \in [0,\delta)$ such that $x_\varepsilon = \max_{t\in[0,\varepsilon)} x(t) < 0$. Hence $x(t) < x_\varepsilon < 0$ for all $t \in [0,\varepsilon)$. By Definition 2.2.1, the state trajectory $x(t)$ satisfies

$$\begin{aligned}x(t) &= x(0) + \int_0^t -(\tau^{-1}+1)x(\tau) + \tau^{-0.25}\,\mathrm{d}\tau \\ &> x(0) + \int_0^t -(\tau^{-1}+1)x_\varepsilon\,\mathrm{d}\tau + \int_0^t \tau^{-0.25}\,\mathrm{d}\tau \\ &= \infty\end{aligned}$$

for all $t \in (0,\varepsilon)$. This contradicts the existence of solution x on $[0,\delta)$.

Note that if $x(0) = 0$, the solution of (3.14) with $p = 2$ exists on $[0,1)$ and it is given by

$$x(t) = \mathrm{e}^{(-\ln(t)-t)} \int_0^t \mathrm{e}^{(\ln(\tau)+\tau)}\tau^{-0.25}\mathrm{d}\tau \qquad \forall t \in [0,1). \tag{3.15}$$

On the other hand, when $p = 1$, for every $x(0) \in \mathbb{R}$, the solution of (3.14) exists on $[0,1)$ and it is given by

$$x(t) = \mathrm{e}^{\left(-t^{0.5}-t\right)}\left(x(0) + \int_0^t \mathrm{e}^{\left(\tau^{0.5}+\tau\right)}\tau^{-0.25}\mathrm{d}\tau\right) \qquad \forall t \in [0,1).$$

The above results shows that the plant **P** as in (3.12) with input u as in (3.13) does not have a solution on any interval of the type $[0,\delta)$ when $p=2$ and $x(0)\neq 0$. It has a unique solution when $p=1$ which can also be concluded from Theorem 3.1.2 since it satisfies both **(A1)** and **(A2)**. If $p\geq 2$, we can always find an L^2 input u such that (3.12) does not have a solution $x(t)$ on any interval of the type $[0,\delta)$, $\delta>0$, for example, $u(t)=t^{-\frac{1}{2p}}$, $t\in[0,1)$.

Now consider plant **P** described by

$$\dot{x} \;=\; \begin{cases} -x+\mathrm{sat}(u) & \forall x\in[-1,1], \\ x-2+\mathrm{sat}(u) & \forall x\in(1,\infty), \\ x+2+\mathrm{sat}(u) & \forall x\in(-\infty,-1), \end{cases} \tag{3.16}$$

$$y \;=\; x, \tag{3.17}$$

where $x(t),u(t),y(t)\in\mathbb{R}$, $\mathrm{sat}:\mathbb{R}\rightarrow\mathbb{R}$ is a saturation function defined by $\mathrm{sat}(u)=u$ for all $u\in(-1,1)$ and $\mathrm{sat}(u)=u/|u|$ otherwise. For the plant **P** as in (3.16), (3.17) and for every initial condition $x(0)\in\overline{\mathbf{B}}_1$, it can be checked that every $u\in L^2(\mathbb{R}_+,\mathbb{R})$ implies the existence of a unique solution and the corresponding output $y\in L^2(\mathbb{R}_+,\mathbb{R})$. But when $x(0)$ is outside the ball $\overline{\mathbf{B}}_3$, i.e., $x(0)\in\mathbb{R}\backslash\overline{\mathbf{B}}_3$, $u\in L^2\not\Rightarrow y\in L^2$ and $\lim_{t\rightarrow\infty}\|y(t)\|=\infty$.

The concept of L^2-stability is originally defined for mapping, see, for example, Vidyasagar [59, Chapter 6.3] or van der Schaft [58, Chapter 1.2]. Its generalization to state equations often overlooks the influence of the initial state on the output (for example in [59, Chapter 6.3]) or the existence of solution of the state equation (for example in [58, Remark 3.1.4] or in [29, Lemma 6.5]). Example (3.12) with $p\geq 2$ shows that the system having a finite L^2-gain (in the sense of [58, Definition 3.1.3]) does not imply L^2-stability. Example (3.16) shows that for every initial condition $x(0)$ in a compact set, every L^2 input u implies the existence of a unique solution to the system equations and the corresponding output y is in L^2, but this property does not hold anymore when the initial condition $x(0)$ is outside the set.

A good definition of L^2-stability for state equations is given in [58, Definition 1.2.11] but it omits the boundedness of the state trajectories. This omission allows an LTI system to be categorized as an L^2-stable system (in the sense of [58, Definition 1.2.11]) but the state grows unbounded for any L^2 input, for example, the plant **P** given by

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}1 & 1\\ 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}0\\ 1\end{bmatrix}u, \quad y=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}. \tag{3.18}$$

In this section, we want to refine again the concept of L^2-stability for dynamical systems which combines the L^2-stability concept from van der Schaft [58] or Vidyasagar [59] with the concept of system stability for linear systems as defined in Curtain [8].

Definition 3.2.1 The plant **P** described by (2.3) is L^2 *system-stable* if for every $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ and $x(0) \in \mathbb{R}^n$, there exists a unique solution x of (2.3) on $\mathbb{R}_+$, the state trajectory x is bounded and the output function y is in $L^2(\mathbb{R}_+, \mathbb{R}^m)$.

It follows that any plant **P** satisfying the assumptions in Theorem 3.1.2 is L^2 system-stable, while the plant **P** in (3.12) with $p \geq 2$, the plant **P** in (3.16),(3.17) and the plant **P** in (3.18) are not L^2 system-stable. Note that if a plant **P** is L^2 system-stable then it is also L^2-stable.

Proposition 3.2.2 *Let the plant* **P** *be defined by* (2.3) *and assume* **(A1)-(A2)**. *Assume that* **P** *has a storage function* H *such that* $H(x) > 0$ *for* $x \neq 0$, $H(0) = 0$, H *is proper and* (2.10) *(strict output passivity) holds with* $k > 0$. *Then* **P** *is* L^2 *system-stable.*

PROOF. Let $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$. It follows from the first part of the proof in Theorem 3.1.2 that for any initial conditions $x(0) \in \mathbb{R}^n$ there exists a global solution x of (2.3) and the state trajectory x is bounded.

By the strict output passivity of **P**, we have

$$\|y\|_{L^2} \leq \frac{1}{k}\|u\|_{L^2} + \sqrt{\frac{2}{k}H(x(0))}.$$

Thus $y \in L^2(\mathbb{R}_+, \mathbb{R}^m)$. □

Corollary 3.2.3 *Let the plant* **P** *be as in Proposition 3.2.2. Then for every* $x(0) \in \mathbb{R}^n$ *and for every* $u \in L^2_{loc}(\mathbb{R}_+, \mathbb{R}^m)$ *there exists a global unique solution of* (2.3).

Remark 3.2.4 A passive system with a proper storage function and satisfying **(A1)–(A2)**, does not necessarily have a global solution for any input $u \in L^2$. Indeed, let the plant **P** be given by

$$\dot{x} = (1-x)^2 u \qquad y = x(1-x)^2,$$

where $x(t), u(t), y(t) \in \mathbb{R}$, with the proper storage function $H = \frac{1}{2}x^2$. **P** is passive, since $\dot{H} = \langle y, u \rangle$. Note that the above system equations satisfy **(A1)–(A2)**. Suppose that the input u is given by

$$u(t) = \begin{cases} -2 & \forall t \in [0,1) \\ 0 & \text{elsewhere,} \end{cases}$$

so that $u \in L^2(\mathbb{R}_+, \mathbb{R})$ and consider the initial condition $x(0) = 0.5$. Then the solution of the differential equation is $x(t) = 1 - (2-2t)^{-1}$, which is defined only on $[0,1)$ and $\lim_{t \to 1} x(t) = -\infty$. □

Remark 3.2.5 It follows from Proposition 3.2.2 and Remark 3.1.1 that if a strictly output passive plant $\mathbf{P}$ is described by (2.5) with a proper storage function H, then $\mathbf{P}$ is L^2 system-stable. Moreover, if $\mathbf{P}$ is zero-state detectable, then for every $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ and for every initial conditions $x(0) \in \mathbb{R}^n$, $x(t) \to 0$ as $t \to \infty$ by Theorem 3.1.2. $\square$

Rapaport and Astolfi [43] proposed the concept of practical L^2-gain which extend the standard notion of L^2-gain by allowing the gain to be a function of the state instead of a constant. Hence the practical L^2-gain allows a tighter gain than the standard L^2-gain when an input with small energy is applied. However, it is already assumed in [43] that solution of the state equations exists for every L^2 input. But since the system equations considered in [43] satisfies **(A1)-(A2)**, it can be shown easily that there exists a unique solution of the state equations for every L^2 input using a similar technique as in Section 3.1.

The concept of integral input-to-state stability (iISS) proposed in Angeli *et al* [2] is related to L^2 system-stability. An iISS system has a property that the state trajectory with input $u \in L^\infty$ is bounded if the input u has a finite energy, e.g., $u \in L^2 \cap L^\infty$. In order to find the relation between iISS and L^2 system-stability, we have to extend the result of iISS for any input $u \in L^2$.

We need the following terminologies before stating the main result in Angeli *et al* [2]. We denote by $\mathcal{K}$ the class of functions $g : \mathbb{R}_+ \to \mathbb{R}_+$ which are zero at zero, strictly increasing and continuous. The set $\mathcal{K}_\infty$ denotes the set of functions $g : \mathbb{R}_+ \to \mathbb{R}_+$ such that $g \in \mathcal{K}$ and $\lim_{x\to\infty} g(x) = \infty$. The set $\mathcal{KL}$ is the class of functions $g : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ such that for any $s \in \mathbb{R}_+$, $g(\cdot, s) \in \mathcal{K}$ and $g(s, \cdot)$ is continuous, decreasing and $\lim\limits_{t\to\infty} g(s,t) = 0$.

Definition 3.2.6 [2] The system $\mathbf{P}$ given by (2.3) is called *integral input-to-state stable* (iISS) if there exist functions $\beta \in \mathcal{KL}$ and $\alpha, \gamma \in \mathcal{K}$ such that for any $x(0) \in \mathbb{R}^n$ and for any $u \in L^\infty_{loc}(\mathbb{R}_+, \mathbb{R}^m)$, there exists a global solution of (2.3) and

$$\|x(t)\| \le \beta(\|x(0)\|, t) + \alpha\left(\int_0^t \gamma(\|u(s)\|)\mathrm{d}s\right)$$

holds for all $t \ge 0$.

Definition 3.2.7 [2] A continuously differentiable function $H : \mathbb{R}^n \to \mathbb{R}_+$ is called an iISS-*Lyapunov function* for system $\mathbf{P}$ given by (2.3) if there exist functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty, \sigma \in \mathcal{K}$ and a continuous positive definite function α_3, such that

$$\alpha_1(\|\xi\|) \le H(\xi) \le \alpha_2(\|\xi\|) \tag{3.19}$$

for all $\xi \in \mathbb{R}^n$, and

$$\frac{\partial H(\xi)}{\partial \xi} f(\xi,\mu) \leq -\alpha_3(\|\xi\|) + \sigma(\|\mu\|) \tag{3.20}$$

for all $\xi \in \mathbb{R}^n$ and all $\mu \in \mathbb{R}^m$.

Note that the Lyapunov function H satisfying (3.19) is proper.

Proposition 3.2.8 [2] *Let the plant* **P** *be given by* (2.3)*. Then* **P** *is iISS if and only if it admits a smooth iISS-Lyapunov function.*

We refer to [2] for the proof of Proposition 3.2.8. The following proposition relates the concept of iISS with that of L^2 system-stability.

Proposition 3.2.9 *Let the plant* **P** *given by* (2.3) *be iISS and assume* **(A1)–(A2)** *hold. Let $H : \mathbb{R}^n \to \mathbb{R}_+$ be a smooth iISS-Lyapunov function such that* (3.19) *and* (3.20) *hold with functions $\alpha_1, \alpha_2 \in \mathcal{K}_\infty, \sigma \in \mathcal{K}$ and a continuous positive definite function α_3. Suppose that the output function h is chosen such that $\|h(\xi)\|^2 \leq \alpha_3(\xi)$ for all $\xi \in \mathbb{R}^n$ and assume there exists $k > 0$ such that $\sigma(\|\mu\|) \leq k\|\mu\|^2$ for all $\mu \in \mathbb{R}^m$. Then* **P** *is L^2 system-stable.*

PROOF. Let $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$. Using the same approach as in the first part of the proof in Theorem and 3.1.2 we obtain that for any initial value $x(0) \in \mathbb{R}^n$, there exists $\delta > 0$ and a unique solution of (2.3) with input $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ on $\mathcal{I} = [0, \delta)$. In particular, x is absolutely continuous on $\mathcal{I}$.

Consider a Lyapunov function $V(x,t) = H(x) + (k+1)\int_t^\infty \|u(\lambda)\|^2 \,\mathrm{d}\lambda$. Using (3.20) and using the assumptions of the proposition we obtain that

$$\begin{aligned} \dot{V} &\leq -\alpha_3(\|x(t)\|) + \sigma(\|u(t)\|) - (k+1)\|u(t)\|^2 \\ &\leq -\|h(x(t))\|^2 - \|u(t)\|^2 \end{aligned} \tag{3.21}$$

holds for all $t \in \mathcal{I}$.

Since $V(x(t),t)$ is absolutely continuous on $\mathcal{I}$ and bounded from below, (3.21) implies that $V(x(t),t)$ is bounded and non-increasing for all $t \in \mathcal{I}$. In particular, the state $x(t)$ never leaves the compact set $\{x \in \mathbb{R}^n \mid H(x) \leq V(x(0),0)\}$ for all $t \in \mathcal{I}$. This implies that $\mathcal{I} = \mathbb{R}_+$ by Corollary 2.2.3.

We will prove that $y \in L^2(\mathbb{R}_+, \mathbb{R}^m)$. It is easy to see that using (3.20) and using the assumptions of the proposition, the function H satisfies

$$\dot{H} \leq -\|h(x(t))\|^2 + k\|u(t)\|^2$$

for all $t \in \mathbb{R}_+$. By integrating the above inequality from $t = 0$ to $t = \infty$ it follows that $y \in L^2(\mathbb{R}_+, \mathbb{R}^m)$. □

3.3 System stable interconnections

The motivation to study L^2 system-stable is analogous to the study of Input-to-State Stability (ISS). By definition, for an ISS system with input u and state x, any input $u \in L^\infty$ implies that there exists a global solution x of the state equation and $x \in L^\infty$. If we define an output y which depends continuously on the state x, then it follows that $u \in L^\infty \Rightarrow y \in L^\infty$. In the same manner, an L^2 system-stable with input u, state x and output y has the property that any input $u \in L^2$ implies that there exists a global solution x of the state equation, $x \in L^\infty$ and $y \in L^2$.

A cascade connection of two ISS systems retains the ISS property of the two systems. The same consequence also applies to the cascade connection of two plants which are L^2 system-stable. Let the plants $\mathbf{P}_i$, $i = 1,2$, be given by

$$\dot{x}_i = f_i(x_i, u_i), \qquad y_i = h_i(x_i) \tag{3.22}$$

where $x_i(t) \in \mathbb{R}^{n_i}$ and $u_i(t), y_i(t) \in \mathbb{R}^{m_i}$. Consider $m_1 = m_2$ and let $\mathbf{P}_1$, $\mathbf{P}_2$ be L^2 system-stable connected by $u_2 = y_1$. Then the whole system with input u_1, state $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ and output y_2 is L^2 system-stable. Indeed, by L^2 system-stability of $\mathbf{P}_1$, any $u_1 \in L^2$ implies the global solution of x_1, and we have $x_1 \in L^\infty$ and $y_1 \in L^2$. Since $u_2 = y_1 \in L^2$, by L^2 system-stability of $\mathbf{P}_2$, there exists global solution of x_2, and we have $x_2 \in L^\infty$ and $y_2 \in L^2$.

The feedback interconnection of ISS systems preserves the ISS property of the closed loop system provided that a small-gain type condition is satisfied (see Jiang *et al* [27] for details). The feedback interconnection version for L^2 system-stable is given in the following proposition.

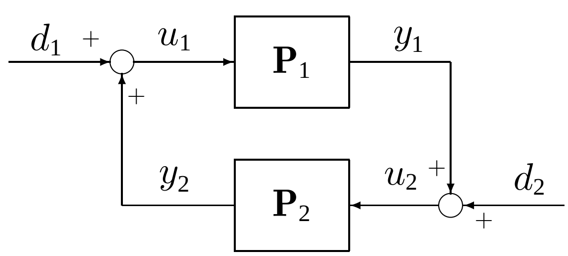


Figure 3.2: The feedback interconnection of systems stable $\mathbf{P}_1$ and $\mathbf{P}_2$.

Proposition 3.3.1 *Let the plants $\mathbf{P}_i$, $i = 1,2$, be given by (3.22) with $m_1 = m_2$. Suppose that for each $i = 1,2$, f_i assumes* **(A1)-(A2)** *and $\mathbf{P}_i$ is L^2 system-stable. Assume that for each $i = 1,2$, $\mathbf{P}_i$ has a finite L^2-gain denoted by γ_i. Suppose that $\mathbf{P}_1$ and $\mathbf{P}_2$ are feedback interconnected as in Figure 3.2 such that $u_1 = d_1 + y_2$ and $u_2 = d_2 + y_1$ where d_1, d_2*

are external signals. If $\gamma_1\gamma_2 < 1$ then the closed-loop system with input $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ and output $\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right]$ is L^2 system-stable.

PROOF. Let $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in L^2(\mathbb{R}_+, \mathbb{R}^{m_1+m_2})$. The closed-loop system $\mathbf{L}$ is given by the following state equation

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, h_2(x_2) + d_1) \\ \dot{x}_2 &= f_2(x_1, h_1(x_1) + d_2). \end{aligned} \tag{3.23}$$

Using **(A1)**, we have that for every compact set $\mathcal{B} \subset \mathbb{R}^{n_1+n_2}$ there exist constants $c_1, c_2 > 0$ such that (3.3) holds for the closed-loop system $\mathbf{L}$. By denoting $\alpha(t) = c_1 + c_2 \left\| \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix} \right\|^2$ and since $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in L^2(\mathbb{R}_+, \mathbb{R}^{m_1+m_2})$, it is easy to see that α is locally integrable and satisfies the condition **(S1)** in Theorem 2.2.2.

Using the assumption **(A2)**, we have that for each fixed $a \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, there exist constants $c_3, c_4 > 0$ such that (3.4) holds for $\mathbf{L}$. By denoting $\beta(t) = c_3 + c_4 \left\| \begin{bmatrix} d_1(t) \\ d_2(t) \end{bmatrix} \right\|^2$ and since $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in L^2(\mathbb{R}_+, \mathbb{R}^{m_1+m_2})$, β is locally integrable and satisfies the condition **(S2)** in Theorem 2.2.2 for the state equation (3.23).

It follows from Theorem 2.2.2 (with α, β as above) that for any initial value $\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} \in \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$, there exists a maximal interval time of definition $\delta > 0$ and a unique solution of (3.23) with input $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ on $\mathcal{J} = [0, \delta)$. In particular, $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ is absolutely continuous on $\mathcal{J}$.

For any measurable function f defined on $\mathcal{J}$, we denote by $\|f\|_{L^2(\mathcal{J})} = (\int_0^\delta \|f(t)\|^2 \mathrm{d}t)^{\frac{1}{2}}$. Using the finite L^2 gain of $\mathbf{P}_1$ and $\mathbf{P}_2$ in the interval time of definition $\mathcal{J}$, we have

$$\begin{aligned} \|y_1\|_{L^2(\mathcal{J})} &\leq \gamma_1 \|d_1 + y_2\|_{L^2(\mathcal{J})} + \beta_1 \\ \|y_2\|_{L^2(\mathcal{J})} &\leq \gamma_2 \|d_2 + y_1\|_{L^2(\mathcal{J})} + \beta_2 \end{aligned}$$

where $\beta_1, \beta_2 \in \mathbb{R}$. By simple algebraic manipulation, it can be shown that

$$\begin{aligned} \|y_1\|_{L^2(\mathcal{J})} &\leq \frac{1}{1-\gamma_1\gamma_2} \Big(\gamma_1 \|d_1\|_{L^2(\mathcal{J})} + \gamma_1\gamma_2 \|d_2\|_{L^2(\mathcal{J})} + \beta_1 + \gamma_1\beta_2 \Big) \\ \|y_2\|_{L^2(\mathcal{J})} &\leq \frac{1}{1-\gamma_1\gamma_2} \Big(\gamma_1\gamma_2 \|d_1\|_{L^2(\mathcal{J})} + \gamma_2 \|d_2\|_{L^2(\mathcal{J})} + \gamma_2\beta_1 + \beta_2 \Big). \end{aligned}$$

Since $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix} \in L^2(\mathbb{R}_+, \mathbb{R}^{m_1+m_2})$, it implies that $\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] \in L^2(\mathcal{J}, \mathbb{R}^{m_1+m_2})$. It follows also that $\left[\begin{smallmatrix} u_1 \\ u_2 \end{smallmatrix}\right] \in L^2(\mathcal{J}, \mathbb{R}^{m_1+m_2})$. By the L^2 system-stability of $\mathbf{P}_1$ and $\mathbf{P}_2$, x_1 and x_2 is bounded on $\mathcal{J}$. Using Corollary 2.2.3 we conclude that the maximal interval of definition of $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ is $\mathbb{R}_+$. Hence the state trajectory $\left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ is bounded on $\mathbb{R}_+$ and $\left[\begin{smallmatrix} y_1 \\ y_2 \end{smallmatrix}\right] \in L^2(\mathbb{R}_+, \mathbb{R}^{m_1+m_2})$. $\square$

3.4 Existence of global solution for time-varying passive systems

Consider the time-varying passive systems **P** given by the state equation

$$\dot{x} = f(t,x,u) \qquad y = h(t,x,u), \tag{3.24}$$

where $x(t) \in \mathbb{R}^n$, $u(t), y(t) \in \mathbb{R}^m$, $f \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ and $h \in \mathcal{C}(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^m)$. We assume that **P** has a time-varying storage function $H \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}_+)$ such that $H(t,x)$ satisfies

$$\dot{H} = \frac{\partial H}{\partial t} + \frac{\partial H}{\partial x} f(t,x,u) \leq \langle y,u \rangle - k\|y\|^2, \tag{3.25}$$

where $k > 0$, i.e., **P** is a time-varying strictly output passive system.

The time-varying storage function H is called *proper* if for every $c > 0$ and for every $t \geq 0$ the set $\{x \in \mathbb{R}^n \mid H(t,x) \leq c\}$ is compact.

We cannot know if this system of differential equations has a global solution for every initial state $x(0)$ and every input function $u \in L^2(\mathbb{R}_+, \mathbb{R}^n)$, and even if it does, we do not know if the solution is unique. Therefore we shall need the following conditions on the system equations f to ensure the existence of global unique solution:

(A3) For every compact set $\mathcal{B} \subset \mathbb{R}^n$, there exist constants $c_1 > 0$ and a locally integrable function $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|f(t,x_1,u) - f(t,x_2,u)\| \leq (c_1\|u\|^2 + \tau(t))\|x_1 - x_2\|, \tag{3.26}$$

for all $u \in \mathbb{R}^m$, $x_1, x_2 \in \mathcal{B}$ and for almost every $t \in \mathbb{R}_+$.

(A4) For each fixed $a \in \mathbb{R}^n$, there exist constants $c_2 > 0$ and a locally integrable function $\upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|f(t,a,u)\| \leq c_2\|u\|^2 + \upsilon(t), \tag{3.27}$$

for all $u \in \mathbb{R}^m$ and for almost every $t \in \mathbb{R}_+$.

Proposition 3.4.1 *Let the plant* **P** *be defined by* (3.24) *and assume* **(A3)-(A4)** *holds. Assume that* **P** *has a proper time-varying storage function H and* (3.25) *holds with $k > 0$.*

Then for every initial condition $x(0) \in \mathbb{R}^n$ and for every $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, the state trajectory x of **P** *is defined uniquely for all $t \geq 0$ and it is bounded.*

PROOF. Using **(A3)**, we have that for every compact set $\mathcal{B} \subset \mathbb{R}^n$ there exist constants $c_1 > 0$ and a locally integrable function $\tau : \mathbb{R}_+ \to \mathbb{R}_+$ such that (3.26) holds. By denoting $\alpha(t) = c_1\|u(t)\|^2 + \tau(t)$ and since $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, it is easy to see that α is locally integrable and satisfies the condition **(S1)** in Theorem 2.2.2.

Using assumption **(A4)**, we have that for each fixed $a \in \mathbb{R}^n$, there exist constants $c_2 > 0$ and a locally integrable function $\upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ such that (3.27) holds. By denoting $\beta(t) = c_2\|u(t)\|^2 + \upsilon(t)$ and since $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, β is locally integrable and satisfies the condition **(S2)** in Theorem 2.2.2 for the state equation (3.24).

It follows from Theorem 2.2.2 (with α, β as above) that for any initial value $x(0) \in \mathbb{R}^n$, there exists $\delta > 0$ and a unique solution of (3.24) with input $u \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ on $\mathcal{I} = [0, \delta)$. In particular, x is absolutely continuous on $\mathcal{I}$.

Consider the storage function $V(t,x) = H(t,x) + \gamma \int_t^\infty \|u(\lambda)\|^2 \mathrm{d}\lambda$ where $\gamma > 1/4k$. Using (3.25) we obtain that for almost every $t \in \mathcal{I}$,

$$\begin{aligned}
\dot{V}(t,x(t)) &= \dot{V}(t,x(t)) - \gamma\|u(t)\|^2 \\
&\leq \langle y(t), u(t)\rangle - k\|y(t)\|^2 - \gamma\|u(t)\|^2 \\
&\leq (\frac{1}{2\theta} - \gamma)\|u(t)\|^2 + (\frac{\theta}{2} - k)\|y\|^2 \quad \forall \theta > 0.
\end{aligned}$$

By choosing $\theta \in (1/2\gamma, 2k)$, we obtain

$$\dot{V}(t,x(t)) \leq -c_5\|u(t)\|^2 - c_6\|y(t)\|^2 \leq 0, \tag{3.28}$$

where $c_5 = \gamma - \frac{1}{2\theta} > 0$ and $c_6 = k - \frac{\theta}{2} > 0$.

Since $V(t,x(t))$ is absolutely continuous as a function of t and bounded from below, (3.28) implies that $V(t,x(t))$ is bounded and non-increasing for all $t \in \mathcal{I}$. In particular, since H is proper the state $x(t)$ never leaves the compact set $\{x \in \mathbb{R}^n \mid H(t,x) \leq V(0,x(0))\}$ for all $t \in \mathcal{I}$. This implies that $\mathcal{I} = \mathbb{R}_+$ by Corollary 2.2.3. □

In Chapter 6, we shall discuss time-varying passive systems obtained from fully-actuated mechanical systems with the *Slotine-Li* controller.

Chapter 4

Input disturbance rejection using LTI controller

Here, we propose a simple LTI controller to solve the input disturbance rejection problem for passive nonlinear plants, where the disturbance can be decomposed into a finite superposition of sine waves of arbitrary but known frequencies and an L^2 signal. In Section 4.2 we study the disturbance rejection problem where the disturbance is generated by a finite-dimensional LTI exosystem. A necessary and sufficient condition is given for the LTI controller to solve the input disturbance rejection problem. The result is extended in Section 4.3 with an additional L^2 component in the disturbance signal.

4.1 Problem formulation

Consider the feedback system in Figure 4.1, where $r = 0$ and the disturbance d is generated by the exosystems **E**, described by

$$\dot{w} = Sw, \qquad d = C_w w. \tag{4.1}$$

Here, $C_w \in \mathbb{R}^{m\times p}$, $w(t) \in \mathbb{R}^p$ is the exosystem state, $S \in \mathbb{R}^{p\times p}$ has its eigenvalues on the imaginary axis and e^{St} is uniformly bounded for $t \geq 0$. An equivalent way of expressing our assumptions on S is the following: $\sigma(S) \subset i\mathbb{R}$ and all its Jordan blocks are of dimension one (i.e., there are no generalised eigenvectors for S).

Let the passive plant **P** be defined as in (2.3) with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^m$ and output $y(t) \in \mathbb{R}^m$. We choose the controller **C** as follows:

$$\dot{x}_c = Ax_c + Be, \tag{4.2}$$
$$y_c = B^T x_c + De, \tag{4.3}$$

where $x_c(t) \in \mathbb{R}^l$, $l \geq p$, $A \in \mathbb{R}^{l\times l}$, $A^T + A = 0$, $B \in \mathbb{R}^{l\times m}$, (B^T, A) observable, $D = D^T \in \mathbb{R}^{m\times m}$ and $D \geq kI, k > 0$. Let **L** be the closed-loop system as in Figure 4.1, with

$u = d + y_c$ and $e = -y$. Then **L** is described by

$$\dot{x} = f(x, B^T x_c - Dh(x) + d), \tag{4.4}$$

$$\dot{x}_c = Ax_c - Bh(x), \tag{4.5}$$

$$y = h(x). \tag{4.6}$$

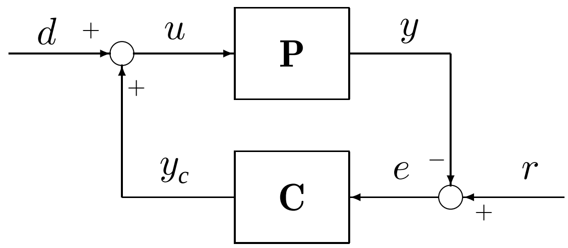


Figure 4.1: The tracking and disturbance rejection problem for the plant **P** and a certain class of signals d and r is to find a controller **C** such that the state trajectories of the closed-loop system **L** are bounded and $e(t) \to 0$ as $t \to \infty$.

Definition 4.1.1 The controller **C** solves the *disturbance rejection problem for the plant* **P** *and the exosystem* **E** *with the initial conditions* $x(0) \in \mathbb{R}^n$, $x_c(0) \in \mathbb{R}^l$ *and* $w(0) \in \mathbb{R}^p$ if, in the closed-loop system **L** shown in Figure 4.1 with $r = 0$ and for d as in (4.1), the state trajectory of the closed-loop system is bounded (hence it has no finite-escape time) and $x(t) \to 0$ as $t \to \infty$ (and hence $y(t) \to 0$ as $t \to \infty$). **C** solves the *disturbance rejection problem for* **P** *and* **E** *locally*, if it solves the disturbance rejection problem for **P** and **E** for all initial conditions $(x(0), x_c(0), w(0))$ in some neighborhood of the origin in $\mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^p$. **C** solves the *disturbance rejection problem for* **P** *and* **E** *globally*, if it solves the disturbance rejection problem for **P** and **E** for any initial conditions $x(0) \in \mathbb{R}^n, x_c(0) \in \mathbb{R}^l$ and $w(0) \in \mathbb{R}^p$.

Definition 4.1.2 Let **P** be as in (2.3) and **C** as in (4.2), (4.3). We say that there is *no pole-zero cancellation between* **P** *and* **C** if for every $x_{co} \in \mathbb{R}^l$, $x_{co} \neq 0$ and for every $x(0) = x_0 \in \mathbb{R}^n$, $u(t) = B^T e^{At} x_{co} \Rightarrow \exists\, t \geq 0$ such that $y(t) \neq 0$.

Note that $y(t)$ exists only as long as the solution x of (2.3) is defined. Thus, if x has a finite-escape time δ, then y is defined on $[0, \delta)$.

Lemma 4.1.3 *Suppose that the plant* **P** *defined by* (2.3) *is zero-state detectable. Let the controller* **C** *be given by* (4.2), (4.3) *and consider the control system* **L** *as in Figure* 4.1, *with* $r = 0$. *If there is no pole-zero cancellation between* **P** *and* **C**, *then* **L** *is zero-state detectable (with input d and output y).*

PROOF. Suppose that $d(t) = 0$ for all $t \geq 0$ and $\left[\begin{smallmatrix} x \\ x_c \end{smallmatrix}\right]$ is a solution of (4.4), (4.5) such that $y(t) = h(x(t)) = 0$ for all $t \geq 0$ where $\left[\begin{smallmatrix} x \\ x_c \end{smallmatrix}\right]$ is defined. Denote $x_0 = x(0) \in \mathbb{R}^n$ and $x_{co} = x_c(0) \in \mathbb{R}^l$. We see from Figure 4.1 that $e(t) = 0$ and hence

$$u(t) = y_c(t) = B^T e^{At} x_{co}.$$

Since there is no-pole zero cancellation between **P** and **C**, this implies that $x_{co} = 0$. Hence $x_c(t) = 0$ for all $t \geq 0$ where $\left[\begin{smallmatrix} x \\ x_c \end{smallmatrix}\right]$ is defined and hence $u(t) = 0$ for such t. This implies that the state trajectory $\left[\begin{smallmatrix} x \\ x_c \end{smallmatrix}\right]$ is defined as long as the solution of

$$\dot{x} = f(x, 0), \qquad x(0) = x_0,$$

is defined. By the zero-state detectability of **P** and since $u(t) = 0$ and $y(t) = 0$ for all $t \geq 0$ where x is defined, it follows that x is defined on $[0, \infty)$ and $x(t) \to 0$ as $t \to \infty$. Hence, $\left[\begin{smallmatrix} x \\ x_c \end{smallmatrix}\right]$ is defined on $[0, \infty)$ and $\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} \to 0$ as $t \to \infty$. □

Remark 4.1.4 Recall from [5], [6], [29] that an *output zeroing submanifold* of the plant **P** is a smooth connected manifold $M \subset \mathbb{R}^n$ containing 0 for which there exists a smooth function $u_z : M \to \mathbb{R}^m$ such that M is invariant along the solutions of $\dot{x} = f(x, u_z(x))$ and the corresponding output of **P** is zero: $h(x) = 0$ for $x \in M$. Then, the property in Definition 4.1.2 shows that for every $x(0) \in M$, the signal $u(t) = u_z(x(t))$, where

$$\dot{x}(t) = f(x(t), u_z(x(t))),$$

cannot be of the form $u(t) = B^T e^{At} x_{co}$ for any $x_{co} \in \mathbb{R}^l$, $x_{co} \neq 0$. The condition in Definition 4.1.2 is related to the *non-resonance condition* for nonlinear systems as described in Marconi *et al* [34]. In particular, if **P** is a linear system, then the condition in Definition 4.1.2 reduces to the usual condition bearing this name from the theory of LTI systems. In particular, this condition follows if **P** has no transmission zeros at the poles of **C**. □

4.2 Disturbances generated by a finite-dimensional LTI exosystem

The next theorem provides necessary and sufficient conditions for the controller **C** in (4.2), (4.3) to solve the disturbance rejection problem locally.

Theorem 4.2.1 *Let the plant* **P** *defined by* (2.3) *be zero-state detectable. Assume that* **P** *has a storage function H such that $H(x) > 0$ for $x \neq 0$, $H(0) = 0$ and* (2.4) *(passivity) holds. Suppose that the exosystem* **E** *generating the disturbance d is given by* (4.1).

Let the controller **C** *be given by* (4.2), (4.3) *and assume that there is no pole-zero cancellation between* **P** *and* **C**. *Then* **C** *solves the disturbance rejection problem for* **P** *and* **E** *locally if and only if there exists a matrix* $\Sigma \in \mathbb{R}^{l \times p}$ *which satisfies*

$$\Sigma S = A\Sigma \quad \text{and} \quad B^T\Sigma + C_w = 0. \tag{4.7}$$

If such a Σ *exists, then it is unique.*

Moreover, if H *is proper, then* **C** *solves the disturbance rejection problem for* **P** *and* **E** *globally.*

PROOF. Suppose that **C** solves the disturbance rejection problem for **P** and **E** locally. Introduce the extended closed-loop system $\tilde{\mathbf{L}}$ which consists of **L** together with the exosystem **E**, with $r = 0$. Let $(x(0), x_c(0), w(0))$ be in a neighborhood of 0 for which the extended closed-loop system has a good behaviour (i.e., x_c is bounded and $x(t) \to 0$). Let $\mathcal{X} \subset \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^p$ be the closure of the corresponding state trajectory of $\tilde{\mathbf{L}}$ so that $\tilde{\mathbf{L}}$ defines a semiflow on $\mathcal{X}$. The trajectory (x, x_c, w) is bounded and according to Proposition 2.3.1 it converges to an invariant set $\boldsymbol{\Omega}$ of the extended closed-loop system. It is easy to see that if $(x_o, x_{co}, w_o) \in \boldsymbol{\Omega}$, then $x_o = 0$. Since (by the assumptions after (2.3)) $f(0, u) = 0 \Leftrightarrow u = 0$ and since $h(0) = 0$, it follows from (2.3), (4.1) and (4.4) that for any trajectory (x, x_c, w) that stays in $\boldsymbol{\Omega}$, denoting (as usual) $u = d + y_c$,

$$x = 0 \Rightarrow u = 0 \Rightarrow B^T x_c + C_w w = 0. \tag{4.8}$$

For such a trajectory, we also have $x_c(t) = e^{At}x_{co}$ (because $e = 0$) and $w(t) = e^{St}w(0)$ for some $x_{co} \in \mathbb{R}^l$. Then, by (4.8),

$$B^T e^{At} x_{co} + C_w e^{St} w(0) = 0. \tag{4.9}$$

We differentiate (4.9) with respect to t, $l-1$ times to conclude that

$$\phi_c x_c(t) + \phi_w w(t) = 0, \tag{4.10}$$

where

$$\phi_c = \begin{bmatrix} B^T \\ B^T A \\ \vdots \\ B^T A^{l-1} \end{bmatrix} \qquad \phi_w = \begin{bmatrix} C_w \\ C_w S \\ \vdots \\ C_w S^{l-1} \end{bmatrix}. \tag{4.11}$$

Since (B^T, A) is observable, ϕ_c has full column rank and $\phi_c^T \phi_c$ is invertible. We multiply (4.10) from the left by ϕ_c^T and conclude that

$$x_c(t) = -[\phi_c^T \phi_c]^{-1} \phi_c^T \phi_w w(t).$$

By taking

$$\Sigma = -[\phi_c^T \phi_c]^{-1} \phi_c^T \phi_w, \tag{4.12}$$

we have $x_c = \Sigma w$ in the invariant set Ω. This, together with (4.1), (4.5) and (4.8) implies that for the trajectory (x, x_c, w) in Ω,

$$\frac{\mathrm{d}}{\mathrm{d}t}(\Sigma w) = \Sigma S w = \dot{x}_c = A\Sigma w, \qquad B^T \Sigma w + C_w w = 0.$$

Since $w(0)$ can be any vector in a neighborhood of zero in $\mathbb{R}^p$, this implies that the equations (4.7) hold.

Conversely, suppose that there exists a matrix Σ satisfying (4.7). Let $\rho = x_c - \Sigma w$, then from (4.1), (4.4) – (4.7) we have

$$\begin{aligned}
u &= B^T[\rho + \Sigma w] - Dh(x) + C_w w \\
&= B^T \rho - Dh(x), &(4.13)\\
\dot{x} &= f(x,u) = f(x, B^T\rho - Dh(x)), &(4.14)\\
\dot{\rho} &= A[\rho + \Sigma w] - Bh(x) - \Sigma \dot{w} \\
&= A\rho - Bh(x), &(4.15)\\
y &= h(x). &(4.16)
\end{aligned}$$

Note that this corresponds to the state equations (4.4) – (4.6) of $\mathbf{L}$, with $d = 0$ and with ρ in place of x_c.

Consider the storage function $H_L(x,\rho) = H(x) + \frac{1}{2}\|\rho\|^2$. Then, using (2.4), (4.14) – (4.16) and $A^T + A = 0$, we obtain that

$$\begin{aligned}
\dot{H}_L &= \frac{\partial H(x)}{\partial x} f\left(x, B^T\rho - Dh(x)\right) + \langle \rho, A\rho - Bh(x)\rangle \\
&\leq \left\langle h(x), B^T\rho - Dh(x)\right\rangle + \langle \rho, A\rho\rangle - \langle \rho, Bh(x)\rangle \\
&= -\langle h(x), Dh(x)\rangle \leq -k\|y\|^2. &(4.17)
\end{aligned}$$

By the assumptions of the theorem and using Lemma 4.1.3, the system described by (4.14) – (4.16) is zero-state detectable. Then, by using the same argument as in the second part of the proof of Proposition 2.4.4, for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\begin{bmatrix} x(0) \\ \rho(0) \end{bmatrix} \in \bar{\mathbf{B}}_\delta \Rightarrow \begin{bmatrix} x(t) \\ \rho(t) \end{bmatrix} \in \bar{\mathbf{B}}_\varepsilon$ for all $t \geq 0$ (hence the solution $\left[\begin{smallmatrix} x \\ \rho \end{smallmatrix}\right]$ is defined on $\mathbb{R}_+$) and $\begin{bmatrix} x(t) \\ \rho(t) \end{bmatrix} \to 0$ as $t \to \infty$. This implies that $x_c - \Sigma w \to 0$ and $y \to 0$. Thus, $\mathbf{C}$ solves the disturbance rejection problem for $\mathbf{P}$ and $\mathbf{E}$ locally.

It can be shown that Σ as in (4.7) is unique. Indeed, suppose that there exists a matrix Σ' which satisfies (4.7), then by multiplying the second identity in (4.7) from the right by $S, S^2 \ldots, S^{l-1}$, we have the following equation

$$\begin{bmatrix} B^T \Sigma' + C_w \\ (B^T \Sigma' + C_w)S \\ \vdots \\ (B^T \Sigma' + C_w)S^{l-1} \end{bmatrix} = 0. \tag{4.18}$$

We use the first identity in (4.7) to transform this into

$$\phi_c \Sigma' + \phi_w = 0,$$

where ϕ_c and ϕ_w are as in (4.11). This implies that Σ' is given by (4.12).

Finally, if H is proper, then H_L is also proper. Using the same argument as in the last part of the proof of Proposition 2.4.4, we see that $\mathbf{C}$ solves the disturbance rejection problem for $\mathbf{P}$ and $\mathbf{E}$ globally. $\square$

Proposition 4.2.2 *Let* $\mathbf{P}$ *and* $\mathbf{C}$ *be as in Theorem 4.2.1, and assume that* $\mathbf{C}$ *solves the disturbance rejection problem for* $\mathbf{P}$ *and* $\mathbf{E}$ *with the initial conditions* $x(0)$, $x_c(0)$ *and* $w(0)$. *Let* x, x_c *and* w *denote the corresponding state trajectories of* $\mathbf{P}$, $\mathbf{C}$ *and* $\mathbf{E}$, *respectively, let* u *be the input function of* $\mathbf{P}$ *and let* y *be its output function (so that* $\lim\limits_{t\to\infty} y(t) = 0$*). Then we also have*

$$\lim_{t\to\infty} u(t) = 0, \quad \lim_{t\to\infty} \dot{u}(t) = 0 \quad \textit{and} \quad \lim_{t\to\infty} \dot{y}(t) = 0.$$

PROOF. As in the first paragraph of the proof of Theorem 4.2.1 we can show that (x, x_c, w) converges to an invariant set Ω of the global closed-loop system, and in Ω we have $u = 0$. Therefore, $\lim\limits_{t\to\infty} u(t) = 0$.

Denoting $\rho = x_c - \Sigma w$ and using (4.14) and (4.16), we have

$$\dot{y} = \frac{\partial h(x)}{\partial x} f\left(x, B^T \rho - Dh(x)\right).$$

We have seen in the proof of Theorem 4.2.1 that $\lim\limits_{t\to\infty} \begin{bmatrix} x(t) \\ \rho(t) \end{bmatrix} = 0$. According to the assumptions used for f and h (including $f \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}^m, \mathbb{R}^n)$ and $f(0,0) = 0$), we obtain that $\lim\limits_{t\to\infty} \dot{y}(t) = 0$.

Using (4.13) and (4.15), we obtain $\dot{u} = B^T(A\rho - By) - D\dot{y}$. Since $\lim\limits_{t\to\infty} \rho(t) = 0$, $\lim\limits_{t\to\infty} y(t) = 0$ and $\lim\limits_{t\to\infty} \dot{y}(t) = 0$, we conclude that $\lim\limits_{t\to\infty} \dot{u}(t) = 0$. $\square$

Remark 4.2.3 In practice, we may assume that

$$(C_w, S) \quad \text{is observable,} \tag{4.19}$$

since otherwise, the exosystem **E** from (4.1) could be replaced by another one of lower order which can generate the same set of signals d.

If (4.7) and (4.19) hold, then Σ has full column rank, i.e., rank $\Sigma = p$. Indeed, otherwise, Ker Σ would be a non-trivial subspace of $\mathbb{R}^p$. It follows from the first identity in (4.7) that Ker Σ is invariant under S. Indeed, if $w_0 \in \operatorname{Ker}\Sigma$, then $\Sigma S w_0 = A\Sigma w_0 = 0$. It follows from the second identity in (4.7) that $\operatorname{Ker}\Sigma \subset \operatorname{Ker} C_w$. Thus, Ker Σ is an unobservable subspace of **E** contradicting (4.19). □

Remark 4.2.4 Note that the condition $\Sigma S = A\Sigma$ with Σ having full column rank implies that $\sigma(S) \subset \sigma(A)$. Indeed, we can find matrices $\tilde{S} = \begin{bmatrix} \tilde{S}_1 \\ \tilde{S}_2 \end{bmatrix} \in \mathbb{R}^{l\times(l-p)}$ and $\tilde{\Sigma} \in \mathbb{R}^{l\times(l-p)}$ such that

$$\begin{bmatrix} \Sigma & \tilde{\Sigma} \end{bmatrix} \begin{bmatrix} S & \tilde{S}_1 \\ 0 & \tilde{S}_2 \end{bmatrix} = A \begin{bmatrix} \Sigma & \tilde{\Sigma} \end{bmatrix}, \tag{4.20}$$

where $\begin{bmatrix} \Sigma & \tilde{\Sigma} \end{bmatrix}$ is invertible. Indeed, after completing Σ to an invertible matrix $\begin{bmatrix} \Sigma & \tilde{\Sigma} \end{bmatrix}$, $\tilde{S}_1$ and $\tilde{S}_2$ are determined by

$$\begin{bmatrix} \tilde{S}_1 \\ \tilde{S}_2 \end{bmatrix} = \begin{bmatrix} \Sigma & \tilde{\Sigma} \end{bmatrix}^{-1} A\tilde{\Sigma}.$$

Therefore,

$$\begin{aligned} & \begin{bmatrix} S & \tilde{S}_1 \\ 0 & \tilde{S}_2 \end{bmatrix} = \begin{bmatrix} \Sigma & \tilde{\Sigma} \end{bmatrix}^{-1} A \begin{bmatrix} \Sigma & \tilde{\Sigma} \end{bmatrix}, \\ \Rightarrow \quad & \sigma\left(\begin{bmatrix} S & \tilde{S}_1 \\ 0 & \tilde{S}_2 \end{bmatrix}\right) = \sigma(A), \\ \Rightarrow \quad & \sigma(S) \subset \sigma(A). \end{aligned}$$

□

Remark 4.2.5 Combining the previous two remarks, we see that if (4.7) and (4.19) hold, then $\sigma(S) \subset \sigma(A)$. □

4.3 An L^2 disturbance is added to the disturbance generated by the exosystem

In this section we revisit the disturbance rejection problem considered in Theorem 4.2.1, but we make it a little harder, by allowing the disturbance d to have an additional component in $L^2(\mathbb{R}_+, \mathbb{R}^m)$.

We need additional assumptions on the function f in (2.3):

(B1) For every compact set $\mathcal{K} \subset \mathbb{R}^n$ and every $i \in \{1,2,\ldots n\}$, there exist constants $c_7, c_8 > 0$ such that

$$\left\| \begin{matrix} \frac{\partial f_i}{\partial x}(x,u) & \frac{\partial f_i}{\partial u}(x,u) \end{matrix} \right\| \leq c_7 + c_8 \|u\|^2 \qquad \forall u \in \mathbb{R}^m, x \in \mathcal{K},$$

where f_i is the i-th component of the vector function f.

(B2) For each fixed $a \in \mathbb{R}^n$, there exist constants $c_9, c_{10} > 0$ such that

$$\|f(a,u)\| \leq c_9 + c_{10}\|u\|^2 \qquad \forall u \in \mathbb{R}^m.$$

Let us consider the feedback system in Figure 4.1, where $r = 0$ and the disturbance d is of the form $d = d_0 + d_E$, where $d_0 \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ and d_E is generated by the exosystem **E** as in (4.1). Our main result in this section states that under these assumptions, the conclusions are the same as in Theorem 4.2.1, that is, d_0 has no effect on the asymptotic behaviour of the closed-loop system.

Theorem 4.3.1 *Let the plant* **P** *defined by* (2.3) *be zero-state detectable and assume* **(B1)-(B2)**. *Assume that* **P** *has a storage function H such that $H(x) > 0$ for $x \neq 0$, $H(0) = 0$, H is proper and* (2.4) *(passivity) holds. Consider the feedback system* **L** *shown in Figure 4.1 with $r = 0$ and $d = d_0 + d_E$, where $d_0 \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ and d_E is generated by* **E** *as in* (4.1). *Let the controller* **C** *be given by* (4.2), (4.3) *and assume that there is no pole-zero cancellation between* **P** *and* **C**.

Then for any initial conditions $x(0) \in \mathbb{R}^n, x_c(0) \in \mathbb{R}^l$ and $w(0) \in \mathbb{R}^p$ and for any $d_0 \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, the state trajectory $\left[\begin{smallmatrix} x \\ x_c \end{smallmatrix}\right]$ of **L** *is bounded (hence there is no finite-escape time) and $x(t) \to 0$ as $t \to \infty$ (and hence $y(t) \to 0$ as $t \to \infty$) if and only if there exists a matrix $\Sigma \in \mathbb{R}^{l \times p}$ which satisfies*

$$\Sigma S = A\Sigma \quad \textit{and} \quad B^T \Sigma + C_w = 0. \tag{4.21}$$

PROOF. The "only if" part follows from Theorem 4.2.1 by taking $d_0 = 0$.

Let $\rho = x_c - \Sigma w$ and $\eta = \left[\begin{smallmatrix} x \\ \rho \end{smallmatrix}\right]$. Using (4.1), (4.4), (4.5), (4.6) and (4.21), we have an extended system **L** given by

$$\dot{\eta} = F(\eta, d_0), \qquad y = h(x), \tag{4.22}$$

where

$$F(\eta, d_0) = \begin{bmatrix} f(x, B^T\rho - Dh(x) + d_0) \\ A\rho - Bh(x) \end{bmatrix}.$$

Note that the above equations are the same as (4.4)–(4.6) with d_0 in place of d and with ρ in place of x_c. By the assumptions of the theorem and using Lemma 4.1.3, the system **L** in (4.22) is zero-state detectable with input d_0 and output y and it is strictly output passive with the proper storage function $H_L(x,\rho) = H(x) + \frac{1}{2}\|\rho\|^2$, i.e., it satisfies $\dot{H}_L \leq \langle y, d_0\rangle - k\|y\|^2$.

Using **(B1)**, it can be shown that for every compact set $\mathcal{K} \subset \mathbb{R}^n \times \mathbb{R}^l$, there exists constants $c_7, c_8 > 0$ such that for every $i \in \{1,2,\ldots n+l\}$,

$$\left\|\frac{\partial F_i(\eta,\tilde{u})}{\partial \eta}\right\| \leq c_7 + c_8\|\tilde{u}\|^2 \qquad \forall \tilde{u} \in \mathbb{R}^m, \eta \in \mathcal{K}, \tag{4.23}$$

where F_i denotes the i-th component of the vector function F. Thus, for every $a \in \mathbb{R}^n \times \mathbb{R}^l$ and for every $c > 0$, by defining $\mathcal{K} = a + \bar{\mathbf{B}}_c$ and using (4.23),

$$\|F_i(\eta_1,\tilde{u}) - F_i(\eta_2,\tilde{u})\| \leq (c_7 + c_8\|\tilde{u}\|^2)\|\eta_1 - \eta_2\| \qquad \forall \eta_1,\eta_2 \in \mathcal{K}, \tilde{u} \in \mathbb{R}^m.$$

It follows that

$$\|F(\eta_1,\tilde{u}) - F(\eta_2,\tilde{u})\| \leq \sqrt{(n+l)}(c_7 + c_8\|\tilde{u}\|^2)\|\eta_1 - \eta_2\| \qquad \forall \eta_1,\eta_2 \in \mathcal{K}, \tilde{u} \in \mathbb{R}^m.$$

Thus, Assumption **(A1)** from Section 3.1 holds for the equations (4.22).

It can be shown that Assumption **(B2)** implies that for each fixed $a \in \mathbb{R}^n \times \mathbb{R}^l$, there exist constants $c_9, c_{10} > 0$ such that

$$\|F(a,\tilde{u})\| \leq c_9 + c_{10}\|\tilde{u}\|^2 \qquad \forall \tilde{u} \in \mathbb{R}^m.$$

This shows that Assumption **(A2)** from Section 3.1 holds for the equations (4.22).

Hence, all the assumptions in Theorem 3.1.2 hold for the system **L** described by (4.22). Using Theorem 3.1.2 for **L** we have that for any initial conditions $\eta(0) \in \mathbb{R}^n \times \mathbb{R}^l$, and for any $d_0 \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, there exists a unique solution η defined for all $t \geq 0$ and $\lim_{t\to\infty} \|\eta(t)\| = 0$, i.e., $\lim_{t\to\infty} \left\| \begin{bmatrix} x(t) \\ x_c(t) - \Sigma w(t) \end{bmatrix} \right\| = 0$. $\square$

4.4 How to construct a controller that solves the disturbance rejection problem

We have seen in Theorem 4.2.1 that the linear controller **C** described by (4.2)–(4.3) solves the disturbance rejection problem for **P** and **E** locally (and, under a further assumption, globally) if and only if the equations (4.7) have a solution. However, this result by itself does not indicate any practical way to construct **C**. In this section, we give an algorithm for constructing **C** using only the eigenvalues of S as our input data (these correspond to the frequencies of the exosystems **E**). In particular, no information about **P** is needed.

Let $\chi(s) = s^\kappa + a_{\kappa-1}s^{\kappa-1} \ldots + a_1 s + a_0$ be the minimal polynomial of $S \in \mathbb{R}^{p\times p}$, so that

$$S^\kappa + a_{\kappa-1}S^{\kappa-1} + \ldots + a_2 S^2 + a_1 S + a_0 = 0, \tag{4.24}$$

where $a_{\kappa-1}, \ldots, a_0 \geq 0$, $\kappa \leq p$ and χ has only simple zeros, all on $i\mathbb{R}$.

Suppose that $S_{min} \in \mathbb{R}^{\kappa\times\kappa}$ is such that $S_{min} + S_{min}^T = 0$ and its characteristic polynomial is χ. If $0 \in \sigma(S)$, then the simplest choice would be

$$S_{min} = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & \Omega_1 & 0 & \cdots & 0 \\ 0 & 0 & \Omega_2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \Omega_\nu \end{bmatrix}, \tag{4.25}$$

where for each $k = 1, \ldots \nu$, $\Omega_k = \begin{bmatrix} 0 & -\omega_k \\ \omega_k & 0 \end{bmatrix}$ for some $\omega_k \in \mathbb{R}\backslash\{0\}$ and $\omega_k \neq \omega_j$ for $k \neq j$. The set $\sigma(S_{min}) = \sigma(S)$ contains 0 and $\pm i\omega_k$ $(k = 1, \ldots \nu)$ (0 and ω_k are the known frequencies of the disturbance signal). If $0 \notin \sigma(S)$, then we omit the first line and the first column in (4.25), so that $\sigma(S_{min})$ contains only $\pm i\omega_k$.

For $i = 1, \ldots m$, let $\Gamma_i \in \mathbb{R}^{\kappa\times 1}$ be such that (Γ_i^T, S_{min}) is observable (the m vectors Γ_i may be taken equal). The observability matrix of (Γ_i^T, S_{min}) is

$$\mathcal{O}_i = \begin{bmatrix} \Gamma_i^T \\ \Gamma_i^T S_{min} \\ \vdots \\ \Gamma_i^T S_{min}^{\kappa-1} \end{bmatrix}, \tag{4.26}$$

for all $i = 1, \ldots, m$. Note that since $\mathcal{O}_i \in \mathbb{R}^{\kappa\times\kappa}$ has full rank, $\mathcal{O}_i$ is invertible.

Lemma 4.4.1 *Let S_{min} and $\Gamma_i, i = 1, \ldots m$ be as above. Then*

$$\mathcal{O}_i S_{min} \mathcal{O}_i^{-1} = S_{obs}, \qquad i = 1, \ldots m, \tag{4.27}$$

where $\mathcal{O}_i$ is as in (4.26) and

$$S_{obs} = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{\kappa-1} \end{bmatrix}.$$

PROOF. Let $i \in \{1, \ldots m\}$. Since $\mathcal{O}_i$ is invertible, we can rewrite (4.27) as follows

$$\mathcal{O}_i S_{min} = S_{obs} \mathcal{O}_i. \tag{4.28}$$

We prove the lemma by checking the identity in (4.28). Evaluating the right-hand side of (4.28) and using (4.24) (for S_{min}), we obtain

$$\begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{\kappa-1} \end{bmatrix} \begin{bmatrix} \Gamma_i^T \\ \Gamma_i^T S_{min} \\ \vdots \\ \Gamma_i^T S_{min}^{\kappa-1} \end{bmatrix} = \begin{bmatrix} \Gamma_i^T S_{min} \\ \Gamma_i^T S_{min}^2 \\ \vdots \\ \Gamma_i^T(-a_0 - a_1 S_{min} \ldots - a_{\kappa-1} S_{min}^{\kappa-1}) \end{bmatrix}$$

$$= \begin{bmatrix} \Gamma_i^T S_{min} \\ \Gamma_i^T S_{min}^2 \\ \vdots \\ \Gamma_i^T S_{min}^{\kappa} \end{bmatrix} = \mathcal{O}_i S_{min}.$$

□

Consider the controller **C** described in state space as in (4.2)–(4.3):

$$\dot{x}_c = A x_c + B e, \tag{4.29}$$

$$y_c = B^T x_c + D e, \tag{4.30}$$

where $x_c \in \mathbb{R}^{\kappa m}$ (κ is as in (4.24)), $e \in \mathbb{R}^m$, $y_c \in \mathbb{R}^m$, the matrices $A \in \mathbb{R}^{\kappa m \times \kappa m}$ and $B \in \mathbb{R}^{\kappa m \times m}$ are given by

$$A = \begin{bmatrix} S_{min} & 0 & \cdots & 0 \\ 0 & S_{min} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{min} \end{bmatrix}, \qquad B = \begin{bmatrix} \Gamma_1 & 0 & \cdots & 0 \\ 0 & \Gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Gamma_m \end{bmatrix}, \tag{4.31}$$

and $D = D^T \in \mathbb{R}^{m \times m}$, $D \geq kI$, where $k > 0$.

Lemma 4.4.2 *Let the matrices A and B be as in (4.31). Then, the square matrix*

$$\phi_c^r = \begin{bmatrix} B^T \\ B^T A \\ \vdots \\ B^T A^{\kappa-1} \end{bmatrix} \tag{4.32}$$

has full rank equal to κm.

Note that the matrix ϕ_c^r in (4.32) is a submatrix of ϕ_c from (4.10).

PROOF. It can be shown that there exists a permutation matrix $T \in \mathbb{R}^{\kappa m \times \kappa m}$ such that

$$T\phi_c^r = \begin{bmatrix} \mathcal{O}_1 & 0 & \cdots & 0 \\ 0 & \mathcal{O}_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{O}_m \end{bmatrix}.$$

Indeed, if $1 \leq j \leq \kappa$, then the j-th line of T has a 1 in position $(j-1)m+1$. If $\kappa+1 \leq j \leq 2\kappa$, then the j-th line of T has a 1 in position $(j-\kappa-1)m+2$, etc.

Since the observability matrix $\mathcal{O}_i$ has full column rank κ for all $i = 1,\ldots m$, $T\phi_c^r$ has full column rank equal to κm. Since T is invertible, it follows that ϕ_c^r has full column rank equal to κm. □

Theorem 4.4.3 *Suppose that the plant* **P** *defined by* (2.3) *satisfies* (2.4) *(passivity) with a storage function H such that $H(x) > 0$ for $x \neq 0$, $H(0) = 0$. Assume that* **P** *is zero-state detectable. Let* **E** *be the exosystem from* (4.1) *and denote by χ the minimal polynomial of S. Let the controller* **C** *be given by* (4.29) *–* (4.31) *where S_{min} has the characteristic polynomial χ and satisfies $S_{min} + S_{min}^T = 0$. We assume that there is no pole-zero cancellation between* **P** *and* **C**.

Then **C** *solves the disturbance rejection problem for* **P** *and* **E** *locally. Moreover, if H is proper, then* **C** *solves the disturbance rejection problem for* **P** *and* **E** *globally.*

PROOF. The theorem is proved by showing that there exists a mapping Σ satisfying (4.7) and then using Theorem 4.2.1.

Since $\phi_c^r \in \mathbb{R}^{\kappa m \times \kappa m}$ from (4.32) has full rank by Lemma 4.4.2, it is invertible. Let

$$\Sigma = -(\phi_c^r)^{-1}\phi_w^r, \tag{4.33}$$

where ϕ_w^r is a submatrix of ϕ_w from (4.11):

$$\phi_w^r = \begin{bmatrix} C_w \\ C_w S \\ \vdots \\ C_w S^{\kappa-1} \end{bmatrix}. \tag{4.34}$$

We see from (4.33) that

$$0 = \phi_c^r \Sigma + \phi_w^r = \begin{bmatrix} B^T \\ B^T A \\ \vdots \\ B^T A^{\kappa-1} \end{bmatrix} \Sigma + \begin{bmatrix} C_w \\ C_w S \\ \vdots \\ C_w S^{\kappa-1} \end{bmatrix}. \tag{4.35}$$

The first line of (4.35) shows that Σ satisfies the second equation in (4.7).

It remains to prove that Σ also satisfies the first equation in (4.7), i.e. $\Sigma S = A\Sigma$. By using the same permutation matrix T as in the proof of Lemma 4.4.2, Σ can be written as

$$\begin{aligned}\Sigma &= -(T\phi_c^r)^{-1}T\phi_w^r \\ &= -\begin{bmatrix}\mathcal{O}_1^{-1} & 0 & \cdots & 0 \\ 0 & \mathcal{O}_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \mathcal{O}_m^{-1}\end{bmatrix} T\phi_w^r.\end{aligned} \tag{4.36}$$

Denote the matrices $\mathcal{O}_{w,j}$, $j = 1,\dots m$, by

$$\mathcal{O}_{w,j} = \begin{bmatrix} C_{w,j} \\ C_{w,j}S \\ \vdots \\ C_{w,j}S^{\kappa-1}\end{bmatrix}, \qquad \text{where} \quad C_{w,j} = \text{the } j\text{-th row of } C_w.$$

Using $\mathcal{O}_{w,j}$, $j = 1,\dots m$, we have

$$T\phi_w^r = \begin{bmatrix}\mathcal{O}_{w,1} \\ \mathcal{O}_{w,2} \\ \vdots \\ \mathcal{O}_{w,m}\end{bmatrix}, \qquad \text{hence} \quad \Sigma = -\begin{bmatrix}\mathcal{O}_1^{-1}\mathcal{O}_{w,1} \\ \mathcal{O}_2^{-1}\mathcal{O}_{w,2} \\ \vdots \\ \mathcal{O}_m^{-1}\mathcal{O}_{w,m}\end{bmatrix}.$$

Then, using Lemma 4.4.1,

$$A\Sigma = -\begin{bmatrix} S_{min}\mathcal{O}_1^{-1}\mathcal{O}_{w,1} \\ S_{min}\mathcal{O}_2^{-1}\mathcal{O}_{w,2} \\ \vdots \\ S_{min}\mathcal{O}_m^{-1}\mathcal{O}_{w,m}\end{bmatrix} = -\begin{bmatrix}\mathcal{O}_1^{-1}S_{obs}\mathcal{O}_{w,1} \\ \mathcal{O}_2^{-1}S_{obs}\mathcal{O}_{w,2} \\ \vdots \\ \mathcal{O}_m^{-1}S_{obs}\mathcal{O}_{w,m}\end{bmatrix}.$$

By some algebraic manipulations using (4.24), it can be checked that

$$S_{obs}\mathcal{O}_{w,j} = \mathcal{O}_{w,j}S.$$

Hence,

$$A\Sigma = -\begin{bmatrix}\mathcal{O}_1^{-1}\mathcal{O}_{w,1}S \\ \mathcal{O}_2^{-1}\mathcal{O}_{w,2}S \\ \vdots \\ \mathcal{O}_m^{-1}\mathcal{O}_{w,m}S\end{bmatrix} = \Sigma S.$$

This shows that Σ from (4.33) satisfies also the first equation in (4.7). □

Note that we can include an L^2 disturbance in the theorem above when H is proper. The result follows directly from Theorem 4.3.1 where the existence of Σ uses the same reasoning as in the proof of Theorem 4.4.3.

4.5 Examples

Example 4.5.1 Consider again the nonlinear plant **P** from Remark 2.4.2, which corresponds to the circuit in Figure 2.1. Here, V_d is not available for measurement and our objective is to control the voltage V_c such that the current x (which is also the state of this system) should go to zero.

Suppose that V_d is generated by an exosystem **E** described by (4.1) where

$$S = \begin{bmatrix} 0 & 1 \\ -10000\pi^2 & 0 \end{bmatrix}, \qquad C_w = \begin{bmatrix} a & b \end{bmatrix},$$

$w(t) \in \mathbb{R}^2$, $V_d(t) = d(t) \in \mathbb{R}$, $a, b \in \mathbb{R}$, a and b are not both zero (hence (C_w, S) is observable). The exosystem **E** generates a sine wave of the form $V_d(t) = A\sin(\omega t + \theta)$, where $\omega = 100\pi$. The phase θ and amplitude A depend on $w(0)$, a and b which are not known to the designer. The parameters R, L and the function ϕ are also not known (but they satisfy $L > 0$ and $x\phi(x,u) \geq 0$).

We solve this problem using the theory from Section 4, so that the block diagram of the closed-loop system is as in Figure 4.1, with $r = 0$, $d = V_d$, $y_c = -V_c$ and $y = x$.

Following Theorem 4.4.3, we choose the controller **C** given by (4.29)–(4.30), where

$$A = \begin{bmatrix} 0 & -100\pi \\ 100\pi & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$x_c(t) \in \mathbb{R}^2$, $e(t) \in \mathbb{R}$, $y_c(t) \in \mathbb{R}$ and $D > 0$. We denote $u = V_d - V_c$.

Since the storage function for **P** given in (2.9) is proper, by Theorem 4.4.3, the controller **C** solves the disturbance rejection problem for **P** and **E** globally. This means that for any initial condition $(x(0), x_c(0), w(0)) \in \mathbb{R}^1 \times \mathbb{R}^2 \times \mathbb{R}^2$, the state trajectory (x, x_c) of the closed-loop system is bounded and $x(t) \to 0$ as $t \to \infty$. According to Proposition 4.2.2, we also have $\lim_{t\to\infty} u(t) = 0$, $\lim_{t\to\infty} \dot{u}(t) = 0$ and $\lim_{t\to\infty} \dot{y}(t) = 0$. □

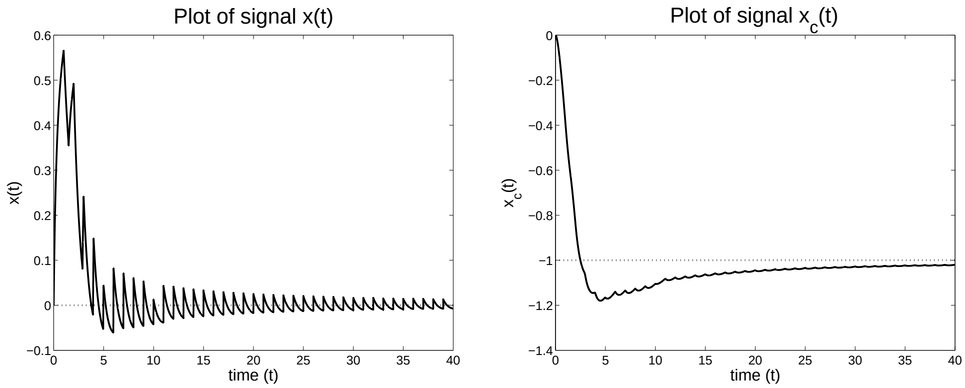
Example 4.5.2 Let the plant **P** and the controller **C** be given by

$$\begin{aligned} \mathbf{P} \ &: \ \dot{x} = -x(1/(u^2+1) + u^2) + u, \quad y = x, \\ \mathbf{C} \ &: \ \dot{x}_c = -y, \qquad y_c = x_c - y, \qquad u = y_c + d, \end{aligned}$$

where $x(t), u(t), y(t) \in \mathbb{R}$. Suppose that $d : \mathbb{R}_+ \to \mathbb{R}$ is given by

$$d(t) = \begin{cases} n & n - \frac{1}{n^2} < t \leq n, \quad n \in \mathbb{N} \\ 1 & \text{elsewhere}, \end{cases} \tag{4.37}$$

so that $d = 1 + d_0$ with $d_0 \in L^2(\mathbb{R}_+, \mathbb{R})$ and $\limsup_{t\in\mathbb{R}_+} |d(t)| = \infty$.



(a) (b)

Figure 4.2: The state trajectory of the closed-loop system in Example 4.5.2 with a constant plus L^2 disturbance. (a) The plot of x. (b) The plot of x_c.

The plant **P** is passive with a proper storage function $H(x) = \frac{1}{2}x^2$ and it is zero-state detectable. Also, it can be checked that there is no pole-zero cancellation between **P** and **C**.

By Theorem 4.3.1, it can be concluded that $\lim_{t\to\infty} \|x(t)\| = 0$. Figure 4.2 shows the convergence of x and x_c with initial conditions $x(0) = 0$ and $x_c(0) = 0$. $\square$

Chapter 5

Tracking of a constant signal for constant incremental passive systems

In many control applications one is interested in operating the system around a *non–zero* equilibrium point. A standard procedure to describe the dynamics in these cases is to generate an *incremental model* whose equilibrium is at zero and with inputs and outputs the deviations with respect to their value at the equilibrium. A natural question that arises is whether a property of the original system is inherited by its incremental model, in particular, the property of passivity.

In this chapter, we study a desirable passivity property around an equilibrium point (x_0, u_0) which generates the output y_0. If a storage function can be found such that the system is dissipative with supply rate $\langle y - y_0, u - u_0 \rangle$, then we can solve the tracking problem for constant signals by recasting it into an input disturbance rejection problem with constant disturbance signal. Thus, the result in Chapter 4 can be applied directly to such system.

Note that one can use the controller design in [5] to solve local output regulation problem for passive nonlinear plant. However, an assumption used in [5] is that the linearization of the system equations at the origin must be stabilizable and detectable. This condition is not satisfied for a large class of passive nonlinear systems. We give an example of electrical circuit which is taken from Section 5.8. Consider the plant **P** given by the state equations:

$$\begin{aligned} \dot{\phi} &= -\left(\frac{1}{2}\ln\left(\frac{1+\phi}{1-\phi}\right)\right)^3 + q^3, \\ \dot{q} &= -\frac{1}{2}\ln\left(\frac{1+\phi}{1-\phi}\right) + u, \\ y &= q^3, \end{aligned}$$

where the state $\begin{bmatrix} \phi(t) \\ q(t) \end{bmatrix} \in (-1,1) \times \mathbb{R}$, the input $u(t) \in \mathbb{R}$ and the output $y(t) \in \mathbb{R}$. Note that for all $\xi \in (-1,1)$, the inverse hyperbolic tangent is given by $\tanh^{-1}(\xi) =$

$\frac{1}{2}\ln\left(\frac{1+\xi}{1-\xi}\right)$. The linearization of **P** at the origin is given by

$$\begin{aligned}\begin{bmatrix}\phi\\ q\end{bmatrix} &= \begin{bmatrix}-\frac{3}{4} & 0\\ -1 & 0\end{bmatrix}\begin{bmatrix}\phi\\ q\end{bmatrix} + \begin{bmatrix}0\\ 1\end{bmatrix}u\\ y &= \begin{bmatrix}0 & 0\end{bmatrix}\begin{bmatrix}\phi\\ q\end{bmatrix}.\end{aligned}$$

It is easy to see that the linearization of **P** at 0 is not detectable. In Section 5.8 it is shown that **P** is constant incremental passive and by using an LTI controller, we are able to track a constant reference signal r and to reject input disturbance d generated by an exosystem for any initial conditions $\begin{bmatrix}\phi(0)\\ q(0)\end{bmatrix} \in (-1,1)\times\mathbb{R}$.

5.1 Constant incremental passive systems

We consider the plant **P** as in (2.3) and the assumptions stated after (2.3). Let $\mathcal{Y}\subset\mathbb{R}^m$ be the range of the output function h containing 0.

Definition 5.1.1 [58] The plant **P** as in (2.3) is said to be *dissipative with respect to the supply rate* $s:\mathbb{R}^m\times\mathbb{R}^m\to\mathbb{R}_+$ if there exists a storage function $H:\mathbb{R}^n\to\mathbb{R}_+$ such that for all $x(0)\in\mathbb{R}^n$, for all $T\geq 0$ and for all admissible input functions $u:[0,T]\to\mathbb{R}^m$ there exists a unique solution of (2.3) on $[0,T]$ and

$$H(x(T))\leq H(x(0))+\int_0^T s(y(t),u(t))\,\mathrm{d}t. \tag{5.1}$$

If the storage function H in the Definition 5.1.1 is $\mathcal{C}^1$, then by dividing both sides in (5.1) by T and by letting $T\to 0$ we have

$$\frac{\partial H(x)}{\partial x}f(x,u)=\dot{H}\leq s(h(x),u), \qquad \forall x\in\mathbb{R}^n, u\in\mathbb{R}^m,$$

where $h(x)=y$.

Definition 5.1.2 The plant **P** is a *constant incremental passive system* if the following is true: If the point $(x_0,u_0)\in\mathbb{R}^n\times\mathbb{R}^m$ is an *equilibrium point* of **P** generating the output $y_0\in\mathcal{Y}$, i.e.,

$$f(x_0,u_0)=0, \quad h(x_0)=y_0, \tag{5.2}$$

then the system is dissipative with respect to the supply rate $s(y,u)=\langle y-y_0,u-u_0\rangle$ and with a storage function $H_{x_0}\in\mathcal{C}^1(\mathbb{R}^n,\mathbb{R}_+)$, i.e.,

$$\dot{H}_{x_0}\leq\langle y-y_0,u-u_0\rangle. \tag{5.3}$$

Definition 5.1.3 The set $\mathcal{R} \subset \mathcal{Y}$ is called *the set of achievable constant reference for* **P** if for any $y_0 \in \mathcal{R}$ there exists an equilibrium point (x_0, u_0) of **P** generating the output y_0 such that (5.2) holds.

The incremental model of **P** around its equilibrium point (x_0, u_0) generating the output y_0 is given by

$$\dot{\tilde{x}} = f(\tilde{x} + x_0, \tilde{u} + u_0), \qquad \tilde{y} = h(\tilde{x} + x_0) - h(x_0), \tag{5.4}$$

where f, h are as in (2.3), the incremental state $\tilde{x} = x - x_0$, the incremental input $\tilde{u} = u - u_0$ and the incremental output $\tilde{y} = y - y_0$ are functions of $t \geq 0$ so that $\tilde{x}(t) \in \mathbb{R}^n$, $\tilde{u}(t), \tilde{y}(t) \in \mathbb{R}^m$. If **P** is constant incremental passive with the corresponding storage function H_{x_0}, it is easy to see that the incremental model of **P** is also passive with respect to storage function $\tilde{H}_{x_0} : \tilde{x} \mapsto H_{x_0}(\tilde{x} + x_0)$, i.e., $\dot{\tilde{H}}_{x_0} \leq \langle \tilde{y}, \tilde{u} \rangle$. (It is the motivation of using the term constant incremental passivity to define the passivity property in (5.3).)

The concept of constant incremental passivity is related to the definition of relative passivity as introduced in Sanders and Verghese [46] where it is used to design controller for switched power converter.

Definition 5.1.4 [46] The plant **P** as in (2.3) is called *relatively passive* if the following is true. If the point $(x_0, u_0) \in \mathbb{R}^n \times \mathbb{R}^m$ is an equilibrium point of **P** generating the output $y_0 \in \mathcal{Y}$ then for all $T \geq 0$ and for all admissible input function $u : [0, T] \to \mathbb{R}^m$ there exists a unique solution of (2.3) on $[0, T]$ with $x(0) = x_0$ such that

$$\int_0^T \langle y(t) - y_0, u(t) - u_0 \rangle \,\mathrm{d}t \geq 0. \tag{5.5}$$

Suppose that **P** is constant incremental passive such that for any equilibrium point (x_0, u_0) we have a storage function H_{x_0} satisfying (5.3). It can be checked that **P** is relatively passive if $H_{x_0}(x_0) = 0$. Indeed, by integrating (5.3) from $t = 0$ to $t = T$ with $x(0) = x_0$ we obtain (5.5).

One particular class of port-controlled Hamiltonian systems, where the storage function H_{x_0} satisfying (5.3) can be constructed, is presented in [22]. For the plant **P**, one of equilibrium points corresponding to $y_0 = 0$ is $(x_0 = 0, u_0 = 0)$.

Remark 5.1.5 The storage function H_{x_0} satisfying (5.3) can be used to show the passivity of **P** in (2.3). Indeed, by having $x_0 = 0$ and $u_0 = 0$, H_0 satisfies $\dot{H}_0 \leq \langle y, u \rangle$. $\square$

Remark 5.1.6 For affine systems described by

$$\dot{x} = f(x) + g(x)u, \qquad y = h(x), \tag{5.6}$$

the condition (5.3) is equivalent to the following conditions

$$\frac{\partial H_{x_0}(x)}{\partial x}[f(x) + g(x)u_0] \leq 0,$$
$$\frac{\partial H_{x_0}(x)}{\partial x} g(x) = h^T(x) - h^T(x_0).$$

Note that by taking $x_0 = 0$ and $u_0 = 0$, the above conditions coincide with the Hill-Moylan's conditions (2.6)–(2.7) as in Remark 2.4.1). $\square$

Remark 5.1.7 Constant incremental passive systems are natural in LTI passive systems. Suppose that an LTI system **P** with state x, input u and output y, is passive with respect to a quadratic storage function H. Let (x_0, u_0) be an equilibrium of **P**, generating the output y_0. Then, the same LTI system **P** is dissipative with respect to the quadratic storage function $H_{x_0} : x \mapsto H(x - x_0)$ and supply rate $\langle y - y_0, u - u_0 \rangle$, i.e., $\dot{H}_{x_0} \leq \langle y - y_0, u - u_0 \rangle$. $\square$

5.2 An example of constant incremental passive systems

Example 5.2.1 Consider the following plant **P**, which is an integrator with a saturated output, namely

$$\dot{x} = u, \qquad y = \tanh(x), \tag{5.7}$$

where $x(t), u(t), y(t) \in \mathbb{R}$ and $\mathcal{R} = (-1, 1)$. It can be checked that for any $y_0 \in \mathcal{R}$, the equilibrium point $(x_0 = \tanh^{-1}(y_0), u_0 = 0)$ and the storage function $H_{x_0}(x) = \int_{x_0}^{x} \tanh(\sigma)\mathrm{d}\sigma - \tanh(x_0)(x - x_0)$ describes the constant incremental passivity of **P**, i.e., it satisfies (5.3). It is easy to see that $H_{x_0}(x) > 0$ for all $x \neq x_0$, $H_{x_0}(x_0) = 0$ and H_{x_0} is also proper.

Example 5.2.2 The nonlinear electrical circuit **P** in Figure 5.1 can be described by

$$\begin{aligned} \dot{I}_2 &= \left(\frac{\mathrm{d}\phi(I_2)}{\mathrm{d}I_2}\right)^{-1} (-\alpha(I_2) + V), \\ \dot{V} &= \left(\frac{\mathrm{d}q(V)}{\mathrm{d}V}\right)^{-1} (-I_2 + I), \qquad (5.8) \\ y &= V, \quad e = y_0 - y, \qquad (5.9) \end{aligned}$$

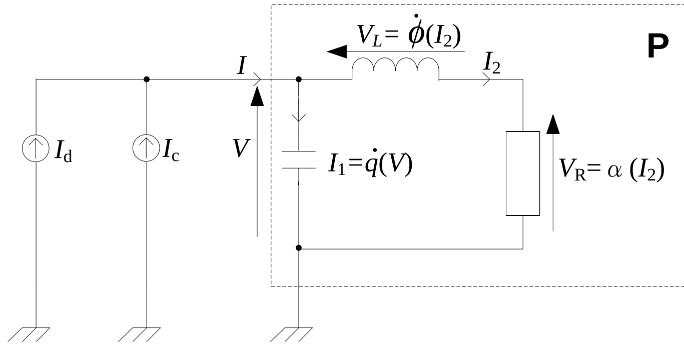


Figure 5.1: Electrical circuit of voltage regulation, where the load consists of a nonlinear resistor $V_R = \alpha(I_2)$, a nonlinear inductor and a nonlinear capacitor.

where $x(t) \equiv \begin{bmatrix} I_2(t) \\ V(t) \end{bmatrix} \in \mathbb{R}^2$ is the state of **P**, $I \in \mathbb{R}$ is the input current, $y \in \mathbb{R}$ is the output voltage. Note that $\mathcal{R} = \mathbb{R}$ in this example. Here, ϕ, q and α are assumed to be $\mathcal{C}^1(\mathbb{R}, \mathbb{R})$, $\phi(0) = 0$, $q(0) = 0$, $\alpha(0) = 0$ and these functions are *strictly monotone increasing*. (The fact that ϕ is strictly monotonically increasing means that $(\phi(a) - \phi(b))(a - b) > 0$ for any $a \neq b$.)

The physical meaning of $\phi(I_2)$ is the magnetic flux of the inductor, so that the voltage across the inductor is $V_L = \dot{\phi}(I_2)$. The meaning of $q(V)$ is the electric charge in the capacitor, so that the current flowing through the capacitor is $I_1 = \dot{q}(V)$. Note that for a linear resistor, $\alpha(I_2) = RI_2$, where $R > 0$ is the resistance, for a linear inductor, $\phi(I_2) = LI_2$, where $L > 0$ is the inductance, and for a linear capacitor, $q(V) = CV$, where $C > 0$ is the capacitance.

Using the storage function

$$H_0(x) \quad = \quad \phi(I_2)I_2 - \int_0^{I_2} \phi(\lambda)\mathrm{d}\lambda + q(V)V - \int_0^V q(\lambda)\mathrm{d}\lambda,$$

It can be shown that **P** is passive with input I and output V, i.e. $\dot{H}_0 \leq \langle V, I \rangle$.

Let $\alpha^{-1}(\cdot)$ denote the inverse function of $\alpha(\cdot)$, such that $\alpha \circ \alpha^{-1}(a) = a$ for any $a \in \mathbb{R}$. It can be checked that for any $y_0 \in \mathcal{R}$, the equilibrium point (x_0, u_0) of **P** is given by

$$x_0 = \begin{bmatrix} \alpha^{-1}(y_0) \\ y_0 \end{bmatrix}, \quad u_0 = \alpha^{-1}(y_0). \tag{5.10}$$

Setting $I_{20} = \alpha^{-1}(y_0)$ and $V_0 = y_0$, the storage function

$$\begin{aligned} H_{x_0}(x) \quad = \quad & \phi(I_2)(I_2 - I_{20})) - \int_{I_{20}}^{I_2} \phi(\lambda)\mathrm{d}\lambda \\ & + q(V)(V - V_0) - \int_{V_0}^{V} q(\lambda)\mathrm{d}\lambda, \end{aligned} \tag{5.11}$$

yields constant incremental passivity of $\mathbf{P}$. Indeed,

$$\begin{aligned}\dot{H}_{x_0}(x) &= \frac{\partial H_{x_0}(x)}{\partial x}\dot{x} = -(I_2 - I_{20})(\alpha(I_2) - \alpha(I_{20})) + \langle V - y_0, I - u_0\rangle \\ &\leq \langle V - y_0, I - u_0\rangle,\end{aligned}$$

where the last inequality is due to the monotonicity of α. It can be checked that for any equilibrium point (x_0, u_0), the function H_{x_0} in (5.11) satisfies $H_{x_0}(x) > 0$ for all $x \neq x_0$ and $H_{x_0}(x_0) = 0$.

Example 5.2.3 Consider a fully-actuated mechanical system $\mathbf{P}$ described by

$$\mathcal{M}(q)\ddot{q} + \mathcal{D}(q,\dot{q})\dot{q} + g(q) = u. \tag{5.12}$$

Such system often originates from Euler-Lagrange equations for mechanical systems and they have been extensively studied, see Astolfi *et al* [3], Ortega *et al* [39], Koivo [30]. Here, $q(t) \in \mathbb{R}^n$ is the vector of *generalized coordinates*, $\mathcal{M}(q)$ is self-adjoint and

$$m_1 I \leq \mathcal{M}(q) \leq m_2 I \qquad \text{where} \quad m_1, m_2 > 0, \tag{5.13}$$

$g(q) = (\nabla V(q))^T$ is a continuous function where $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_+)$ is called the *potential energy*, and $u(t) \in \mathbb{R}^n$ is the input (usually, forces or torques). The function $\mathcal{M}(\cdot)$ is assumed to be continuously differentiable and $\mathcal{D}(\cdot,\cdot)$ is assumed to be continuous. As usual, we denote $\dot{\mathcal{M}}(q,\dot{q}) = \sum_{j=1}^{n} \frac{\partial \mathcal{M}}{\partial q_j}\dot{q}_j$. The state of this system is the vector $x = \left[\begin{smallmatrix} q \\ \dot{q}\end{smallmatrix}\right]$. We assume that $J(q,\dot{q}) = \dot{\mathcal{M}}(q,\dot{q}) - 2\mathcal{D}(q,\dot{q})$ satisfies $J^T(q,\dot{q}) + J(q,\dot{q}) \leq 0$, so that

$$\left\langle \left(\frac{1}{2}\dot{\mathcal{M}} - \mathcal{D}\right) a, a \right\rangle \leq 0 \qquad \forall a \in \mathbb{R}^n. \tag{5.14}$$

Suppose that $V(q)$ is also convex.

The plant $\mathbf{P}$ with output signal $y = \dot{q}$ is *passive* with respect to the storage function $H(x) = \frac{1}{2}\langle \mathcal{M}(q)\dot{q}, \dot{q}\rangle + V(q)$, i.e., $\dot{H} \leq \langle \dot{q}, u\rangle$. Note that $\mathcal{R} = \{0\}$. For any $q_0 \in \mathbb{R}^n$, $(x_0 = \left[\begin{smallmatrix} q_0 \\ 0\end{smallmatrix}\right], u_0 = g(q_0))$ is the equilibrium point of $\mathbf{P}$ with the corresponding output $y_0 = 0$. By the convexity of V, it can be checked that the storage function $H_{x_0}(x) = \frac{1}{2}\langle \mathcal{M}(q)\dot{q}, \dot{q}\rangle + V(q) - (q - q_0)^T g(q_0) - V(q_0)$ satisfies $\dot{H}_{x_0}(x) \leq \langle y - y_0, u - u_0\rangle$ and $H_{x_0}(x) \geq 0$ for all $x \in \mathbb{R}^{2n}$, i.e., the mechanical system $\mathbf{P}$ is a constant incremental passive system.

5.3 Stability of constant incremental passive systems

A simple control law can be assigned to constant incremental passive systems to stabilize a desirable equilibrium state. Suppose that (x_0, u_0) is an equilibrium point of a constant incremental passive system $\mathbf{P}$ with $H_{x_0}(x) > 0$ for all $x \neq x_0$ and $H_{x_0}(x_0) = 0$. If $u(t) = u_0$ for all $t \geq 0$ then the equilibrium state x_0 is Lyapunov stable. This can be shown by using $H_{x_0} : \mathbb{R}^n \to \mathbb{R}_+$ as a Lyapunov function. Indeed, since $u = u_0$, $\dot{H}_{x_0} \leq 0$ and Lyapunov stability theorem shows that x_0 is Lyapunov stable, i.e., for every $\varepsilon > 0$ there exists $\delta > 0$ such that $x(0) \in \bar{\mathbf{B}}_\delta + x_0 \Rightarrow x(t) \in \bar{\mathbf{B}}_\varepsilon + x_0$ for all $t \geq 0$.

Under a suitable detectability condition, the equilibrium state x_0 can be made asymptotically stable by output feedback as summarized in the following proposition.

Proposition 5.3.1 *Consider the plant* $\mathbf{P}$ *described by* (2.3) *and the assumptions stated after* (2.3). *Suppose that* $\mathbf{P}$ *is constant incremental passive. Let* (x_0, u_0) *be the equilibrium point of* $\mathbf{P}$ *with the corresponding output* y_0 *so that there exists storage function* $H_{x_0} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_+)$ *satisfying* (5.3). *Assume that* $H_{x_0}(x) > 0$ *for all* $x \neq x_0$, $H_{x_0}(x_0) = 0$ *and the incremental model of* $\mathbf{P}$ *as in* (5.4) *is zero-state detectable.*

Let the input u *be given by* $u(t) = u_0 - K(y - y_0)$, $K = K^T > kI$, $k > 0$. *Then the equilibrium state* x_0 *of the closed-loop system is asymptotically stable. Moreover, if* H_{x_0} *is proper, then* x_0 *is globally asymptotically stable.*

PROOF. Using $u(t) = u_0 - K(y(t) - y_0)$, the incremental model of the closed-loop system $\mathbf{L}$ is given by

$$\dot{\tilde{x}} = f(\tilde{x} + x_0, -K\tilde{y} + u_0), \qquad \tilde{y} = h(\tilde{x} + x_0) - h(x_0), \tag{5.15}$$

where the incremental state $\tilde{x} = x - x_0$ and the incremental output $\tilde{y} = y - y_0$ are functions of t so that $\tilde{x}(t) \in \mathbb{R}^n$ and $\tilde{y}(t) \in \mathbb{R}^m$.

It is easy to see that $\mathbf{L}$ is zero-state detectable. Indeed, let $\tilde{y}(t) = 0$ for all $t \geq 0$ so that the incremental model of the closed-loop system is described by $\dot{\tilde{x}} = f(\tilde{x} + x_0, u_0)$. Since the incremental model of $\mathbf{P}$ is zero-state detectable, $\dot{\tilde{x}} = f(\tilde{x} + x_0, u_0)$ and $\tilde{y} = 0$ imply that $\|\tilde{x}(t)\| \to 0$ as $t \to \infty$. Hence, for the closed-loop system, we have that if $y(t) - y_0 = 0$ for all $t \geq 0$ it implies that $\|x(t) - x_0\| \to 0$ as $t \to \infty$.

It can be checked that the time derivative of H_{x_0} along the trajectory of the closed-loop system satisfies

$$\dot{H}_{x_0} \leq \langle y - y_0, u - u_0 \rangle \leq -k\|y - y_0\|^2.$$

This implies (using H_{x_0} as a Lyapunov function) that the closed-loop system is stable. It follows that there exists $\delta > 0$ such that $x(0) \in \bar{\mathbf{B}}_\delta + x_0 \Rightarrow x(t) \in \bar{\mathbf{B}}_1 + x_0$ for all $t \geq 0$. According to La-Salle invariance principle, such a state trajectory x converges to the largest invariant set M contained in $\{z \in \bar{\mathbf{B}}_1 + x_0 \mid \dot{H}_{x_0}(z) = 0\}$. In the invariant set M, H_{x_0} is constant along state trajectories and hence $y = y_0$ along such trajectories. By the zero-state detectability of the incremental model of the closed-loop system, all these trajectories converge to x_0, hence $H_{x_0}(z) = H_{x_0}(x_0) = 0$ for all $z \in M$. Since $H(z) > 0$ for all $z \neq x_0$, we obtain $M = \{x_0\}$, so that the equilibrium state x_0 is asymptotically stable.

When H_{x_0} is proper, then every state trajectory of the closed-loop system remains bounded. Thus, for any state trajectory x, we can apply the preceding argument with $\bar{\mathbf{B}}_1$ replaced by a ball $\bar{\mathbf{B}}_\lambda$ that contains this state trajectory. Then, we conclude that $\lim_{t\to\infty} \|x(t) - x_0\| = 0$, as claimed in the proposition. $\square$

For systems described by Euler-Lagrange equations, the feedback control law in Proposition 5.3.1 resembles the energy shaping technique as presented in [39, Proposition 3.1]. The feedback control law in energy shaping technique consists of two components. The first component shapes the potential energy such that the closed-loop system energy has a minimum at the desired equilibrium state and the second component gives the required damping to stabilize the systems. This principle is similar to the output feedback control law $u = u_0 - K(y - y_0)$ applied to the mechanical systems as in Example 5.2.3. The term $u_0 = g(q_0)$ shapes the potential energy in Example 5.2.3 so that we have closed-loop energy H_{x_0} with a global minimum at $\left[\begin{smallmatrix} q_0 \\ 0 \end{smallmatrix}\right]$ and the term $K(y - y_0) = K\dot{q}$ provides the necessary damping.

The drawback of the feedback control law as in Proposition 5.3.1 is that the precise knowledge of the system equations is necessary to derive the constant input u_0. When parameters in the system equations drift from the nominal value, the feedback control law as above may produce tracking error which does not converge to zero.

It will be shown later, in Section 5.7, that if the plant $\mathbf{P}$ is known to be constant incremental passive system then a PI controller can be used to track a constant reference signal. It is robust to the parameter uncertainties as long as the system remains constant incremental passive.

Note that the result of state convergence with L^2 signal from Chapter 3 is still applicable to Proposition 5.3.1 provided that some technical assumptions hold. This is summarised in the following corollary.

Corollary 5.3.2 *Let the plant* $\mathbf{P}$ *be defined by* (2.3) *and assume* **(A1)-(A2)** *as in Chapter 3. Suppose that* $\mathbf{P}$ *is constant incremental passive. Let* (x_0, u_0) *be the equilibrium point of* $\mathbf{P}$ *with the corresponding output* y_0 *so that there exists a proper storage function* $H_{x_0} \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_+)$ *satisfying* (5.3). *Assume that* $H_{x_0}(x) > 0$ *for all* $x \neq x_0$, $H_{x_0}(x_0) = 0$ *and the incremental model of* $\mathbf{P}$ *as in* (5.4) *is zero-state detectable.*

Let the input u *be given by* $u(t) = u_0 - K(y - y_0) + d$, $K = K^T > kI$, $k > 0$ *where* d *is an external disturbance signal. Then for every initial condition* $x(0) \in \mathbb{R}^n$ *and for* *every $d \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, the state trajectory x of* **P** *is defined for all $t \geq 0$ and it satisfies $\|x(t) - x_0\| \to 0$ as $t \to \infty$ (and hence $\|y(t) - y_0\| \to 0$ as $t \to \infty$).*

PROOF. The incremental model of the closed-loop system **L** is given by

$$\dot{\tilde{x}} = f(\tilde{x} + x_0, -K\tilde{y} + u_0 + d), \qquad \tilde{y} = h(\tilde{x} + x_0) - h(x_0),$$

where the incremental state $\tilde{x} = x - x_0$, the disturbance d and the incremental output $\tilde{y} = y - y_0$ are functions of t so that $\tilde{x}(t) \in \mathbb{R}^n$, $d(t) \in \mathbb{R}^m$ and $\tilde{y}(t) \in \mathbb{R}^m$. Using a proper storage function $\tilde{H}_{x_0} : \tilde{x} \mapsto H_{x_0}(\tilde{x} + x_0)$ and using constant incremental passivity of **P**, **L** is strictly output passive with input d and output $\tilde{y}$, i.e.,

$$\dot{\tilde{H}}_{x_0} \leq \langle \tilde{y}, d \rangle - k\|\tilde{y}\|^2.$$

Therefore, **L** satisfies all assumptions in Theorem 3.1.2. Applying Theorem 3.1.2 to the incremental model of the closed-loop system **L**, we have that for any initial condition $\tilde{x}(0) \in \mathbb{R}^n$ and for every $d \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, $\tilde{x}(t) \to 0$ as $t \to \infty$. This implies that for any initial condition $x(0) \in \mathbb{R}^n$ and for every $d \in L^2(\mathbb{R}_+, \mathbb{R}^m)$, $\|x(t) - x_0\| \to 0$ as $t \to \infty$ for the original closed-loop system. □

5.4 Constant incremental passivity of affine nonlinear systems

Suppose that the plant **P** is of the form

$$\dot{x} = f(x) + gu, \qquad y = h(x), \tag{5.16}$$

where x, u, y are functions of t, $x(t) \in \mathbb{R}^n$, $u(t), y(t) \in \mathbb{R}^m$, the functions $f \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^n)$ and $h \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^m)$ are locally Lipschitz and the matrix $g \in \mathbb{R}^{n \times m}$ is constant and has full rank.

Let (x_0, u_0) be the equilibrium of **P** generating the output $y_0 = h(x_0)$. The incremental model of **P** around the equilibrium point (x_0, u_0) is described by

$$\begin{aligned} \dot{\tilde{x}} &= f(\tilde{x} + x_0) - f(x_0) + g\tilde{u} \\ \tilde{y} &= h(\tilde{x} + x_0) - h(x_0), \end{aligned} \tag{5.17}$$

where $\tilde{x} = x - x_0$ is the incremental state, $\tilde{u} = u - u_0$ is the incremental input and $\tilde{y} = y - y_0$ is the incremental output.

Remark 5.4.1 Note that if **P** is as in (5.16) with $m \leq n$ then the equilibrium point (x_0, u_0) can be characterized as follows. The set of equilibrium state is given by $\mathcal{E} =$

$\{\bar{x} \in \mathbb{R}^n \,|\, g^{\perp} f(\bar{x}) = 0\}$ where $g^{\perp} \in \mathbb{R}^{(n-m)\times n}$ is a full-rank left-annihilator of g, i.e., $g^{\perp} g = 0$ and $\operatorname{rank}\{g^{\perp}\} = n - m$. It can be checked that the constant input u_0 and constant output y_0 corresponding to the equilibrium state $x_0 \in \mathcal{E}$ are given by

$$\begin{aligned} u_0 &= (g^T g)^{-1} g^T f(x_0) \\ y_0 &= h(x_0). \end{aligned} \tag{5.18}$$

The set of achievable constant reference $\mathcal{R}$ is given by $\mathcal{R} = \{h(x_0) \in \mathbb{R}^m \,|\, x_0 \in \mathcal{E}\}$. □

Theorem 5.4.2 *Suppose that the plant* **P** *as in* (5.16) *is passive with a convex storage function* $H \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}_+)$. *Let* (x_0, u_0) *be the equilibrium of* **P** *generating the output* y_0. *Assume that the* "constant incremental stability" condition

$$[\nabla H(x) - \nabla H(x_0)]\,[f(x) - f(x_0)] \leq 0 \tag{5.19}$$

holds for all $x \in \mathbb{R}^n$. [1]

Then the incremental model of **P** *as in* (5.17) *is also passive with convex storage function* $\tilde{H}_{x_0} \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}_+)$, *defined by*

$$\tilde{H}_{x_0}(\tilde{x}) = H(\tilde{x} + x_0) - H(x_0) - \nabla H(x_0)\tilde{x}. \tag{5.20}$$

Or, equivalently, **P** *is also dissipative w.r.t. supply rate* $s(y,u) = \langle y - y_0, u - u_0\rangle$ *and with convex storage function* $H_{x_0} \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}_+)$,

$$H_{x_0}(x) = H(x) - H(x_0) - \nabla H(x_0)\,(x - x_0).$$

PROOF. The passivity of **P** implies that the Hill-Moylan's conditions (2.6), (2.7) as in Remark 2.4.1 holds.

To prove the passivity of the incremental model, calculate the time derivative of (5.20) along the state trajectories of (5.17).

$$\begin{aligned} \dot{\tilde{H}}_{x_0}(\tilde{x}) &= (\nabla H(\tilde{x} + x_0) - \nabla H(x_0))\,(f(\tilde{x} + x_0) - f(x_0) + g\tilde{u}) \\ &\leq (\nabla H(\tilde{x} + x_0) - \nabla H(x_0))\,g\tilde{u} \\ &= (h(\tilde{x} + x_0) - h(x_0))^T\,\tilde{u} = \langle \tilde{y}, \tilde{u}\rangle, \end{aligned}$$

where we have used (5.19) to obtain the inequality and used (2.7) and (5.17) to conclude the passivity of the incremental model.

[1]The gradient of a scalar function $H : \mathbb{R}^n \to \mathbb{R}$ is denoted by $\nabla H(x) = \frac{\partial}{\partial x} H(x)$. Here, we treat $\nabla H(x)$ as a row vector, following the general convention for Jacobian matrices.

We now prove that $\tilde{H}_{x_0}(\tilde{x})$ is nonnegative. Using convexity of $H(x)$, we obtain

$$\frac{\partial^2 \tilde{H}_{x_0}(\tilde{x})}{\partial \tilde{x}^2} = \frac{\partial^2 H(x)}{\partial x^2} \geq 0,$$

which shows the convexity of $\tilde{H}_{x_0}(\tilde{x})$. Since $\nabla \tilde{H}_{x_0}(0) = 0$ and $\tilde{H}_{x_0}(\tilde{x})$ is convex, the point 0 is a minimum point of $\tilde{H}_{x_0}(\tilde{x})$. This implies also that $\tilde{H}_{x_0}(\tilde{x}) \geq 0$.

It remains to prove that the passivity of the incremental model of **P** is equivalent to the dissipativity of **P** w.r.t. $\langle y - y_0, u - u_0 \rangle$. Using coordinate changes $\tilde{x} = x - x_0$, $\tilde{u} = u - u_0$ and $\tilde{y} = y - y_0$, we have that $H_{x_0}(\tilde{x} + x_0) = \tilde{H}_{x_0}(\tilde{x})$. Using the passivity of the incremental model of **P**, it is easy to show that

$$\dot{H}_{x_0} = \dot{\tilde{H}}_{x_0} \leq \langle \tilde{y}, \tilde{u} \rangle = \langle y - y_0, u - u_0 \rangle.$$

□

Note that if **P** is as in Theorem 5.4.2 and (5.19) holds for all $x \in \mathbb{R}^n$ then the equilibrium state $\tilde{x} = 0$ of the incremental model of **P** with $\tilde{u} = 0$ is Lyapunov stable. This is the reason of introducting the notion of incremental stability condition for (5.19).

This notion of constant incremental stability differs from the incremental stability given in Angeli [1]. Incremental asymptotic stability in [1] corresponds to the existence of a unique state trajectory $\bar{x}$ for every bounded input function $\bar{u}$ such that for any initial condition $x(0)$, the state trajectory x of the system with input $\bar{u}$ satisfies $\|x(t) - \bar{x}(t)\| \to 0$ as $t \to \infty$. Since the notion of incremental stability in [1] is defined for all bounded input function, it is stronger than the one we used (which only considers constant input). Also, we do not require the convergence of the state trajectories to the equilibrium state, i.e., if (x_0, u_0) is an equilibrium point then if we apply an input $u(t) = u_0$ for all $t \geq 0$ to a constant incremental passive system then the corresponding equilibrium state $x = x_0$ becomes Lyapunov stable, not necessarily asymptotically stable.

Remark 5.4.3 Suppose that the plant **P** is as in Theorem 5.4.2 so that it is passive with a convex storage function $H \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}_+)$. If for all equilibrium points (x_0, u_0) the constant incremental stability condition (5.19) holds for all $x \in \mathbb{R}^n$ then **P** is a constant incremental passive system. □

It is easy to see that the new storage function $\tilde{H}_{x_0}$ as in (5.20) has a minimum at 0 and $\tilde{H}_{x_0}(0) = 0$. However, the point 0 may not be a unique minimum, for example, $\tilde{H}_{x_0} : \mathbb{R} \to \mathbb{R}_+$ constructed as in (5.20) with $x_0 < 0$ from a convex function $H : \mathbb{R} \to \mathbb{R}_+$

$$H(x) = \begin{cases} \frac{1}{2}x^2, & x \geq 0 \\ 0, & x < 0. \end{cases} \tag{5.21}$$

Note that, in the example above, $\tilde{H}_{x_0}$ is not proper. We will show that a strictly convex H is sufficient to ensure that the new storage function $\tilde{H}_{x_0}$ has a unique minimum at 0 and is proper. We need the following two lemmas before we state the result.

Lemma 5.4.4 *Suppose that $W \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ is a strictly convex function, has a minimum at* 0 *and* $W(0) = 0$.

(i) *For every $\kappa > 0$, $W(x) > 0$ for all $x \in \partial \mathbf{B}_\kappa$.*

(ii) *For every $k = 1, 2, \ldots$, the set $\{x \in \mathbb{R}^n \,|\, W(x) \leq k\alpha\}$ is compact.*

PROOF. Note that since W has a minimum at 0, $W(0) = 0$ and is strictly convex, it implies that 0 is a global minimum and $W(x) \geq 0$ for all $x \in \mathbb{R}^n$.

Proof of (i). Take any $\kappa > 0$. We prove the lemma by using contradiction. Since $W(x) \geq 0$ for all $x \in \partial \mathbf{B}_\kappa$, we prove it by checking whether $W(x) = 0$ for some $x \in \partial \mathbf{B}_\kappa$.

Suppose there exists $x \in \partial \mathbf{B}_\kappa$ such that that $W(x) = 0$. Choose $0 < \gamma < \|x\|$ so that $\frac{\gamma}{\|x\|} x \in \mathbf{B}_\kappa \backslash \{0\}$. Then by denoting $\lambda = \frac{\gamma}{\|x\|}$ and by strict convexity of W we obtain

$$W(\lambda x) < \lambda W(x) + (1 - \lambda) W(0).$$

Since $W(x) = 0$ and $W(0) = 0$ we get $W(\lambda x) < 0$, a contradiction.

Proof (ii). Denote by $\alpha = \min_{x \in \partial \mathbf{B}_1} W(x)$. Note that $\alpha > 0$ by Lemma 5.4.4(i).

Take any $k = 1, 2, \ldots$. We prove last part of the lemma by showing that

$$\mathbb{R}^n \backslash \bar{\mathbf{B}}_k \subset \{x \in \mathbb{R}^n \,|\, W(x) > k\alpha\}. \tag{5.22}$$

Indeed, if (5.22) is satisfied then

$$\{x \in \mathbb{R}^n \,|\, W(x) \leq k\alpha\} \subset \bar{\mathbf{B}}_k.$$

Take any point $a \in \mathbb{R}^n \backslash \bar{\mathbf{B}}_k$ and denote by $b_k = \frac{k}{\|a\|} a$. Note that $\frac{k}{\|a\|} < 1$ and $b_k \in \partial \mathbf{B}_k$. It can be shown that $W(b_k) \geq k\alpha$. Indeed, if $k = 1$ then by definition of α, $W(b_k) \geq \alpha$. If $k = 2, 3, \ldots$, by using a contradiction, assume that $W(b_k) < k\alpha$. Then by the strict convexity of W and by denoting $\lambda = \frac{1}{k} < 1$,

$$W(\lambda b_k) < \lambda W(b_k) + (1 - \lambda) W(0) < \alpha,$$

which contradicts the definition of α since $\lambda b_k \in \partial \mathbf{B}_1$.

Using the strict convexity of W and since $\frac{\|a\|}{k} > 1$ we obtain

$$\begin{aligned} & W(b_k) < \frac{k}{\|a\|} W(a) + \left(1 - \frac{k}{\|a\|}\right) W(0) = \frac{k}{\|a\|} W(a) \\ \Leftrightarrow \quad & W(a) > \frac{\|a\|}{k} W(b_k) > k\alpha. \end{aligned}$$

□

Proposition 5.4.5 *Consider a strictly convex storage function $H \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}_+)$. Then for every $x_0 \in \mathbb{R}^n$, the new storage function $\tilde{H}_{x_0} \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R}_+)$ as constructed in* (5.20) *has a global minimum at* 0 *and is proper.*

PROOF. It can be checked that $\tilde{H}_{x_0}$ as in (5.20) is strictly convex and has a global minimum at the origin. Indeed, by using a contradiction, suppose that there exists $a, b \in \mathbb{R}^n$ and $\lambda \in (0,1)$ such that

$$\tilde{H}_{x_0}(\lambda a + (1-\lambda)b) = \lambda \tilde{H}_{x_0}(a) + (1-\lambda)\tilde{H}_{x_0}(b). \tag{5.23}$$

It can be checked that by using (5.20), (5.23) is equivalent to

$$H(\lambda a + (1-\lambda)b) = \lambda H(a) + (1-\lambda)H(b),$$

which contradicts the strict convexity of H. Using a similar approach as in the proof of Theorem 5.4.2, we have that 0 is a minimum point. By the strict convexity of $\tilde{H}_{x_0}$, it is easy to see that 0 is a global minimum.

By using Lemma 5.4.4(ii), it can be shown that for every $c > 0$, the set $\{\tilde{x} \in \mathbb{R}^n \,|\, \tilde{H}_{x_0}(\tilde{x}) \leq c\}$ is compact. □

5.5 A class of port-controlled Hamiltonian systems

In this section we identify a class of port-controlled Hamiltonian systems which belongs to the class of constant incremental passive systems. This includes a large class of electrical circuits with convex electric and magnetic energy functions.

Port-controlled Hamiltonian systems on the state space $\mathbb{R}^n$ are, by definition, systems described by the equations

$$\dot{x} = A(x)\,(\nabla H(x))^T + B_R(x)\, f_R + B(x)u, \tag{5.24}$$

$$e_R = B_R^T(x)\,(\nabla H(x))^T, \tag{5.25}$$

$$y = B^T(x)\,(\nabla H(x))^T, \tag{5.26}$$

$$f_R = -\alpha(e_R), \tag{5.27}$$

where the state x, the input u and the output y are functions of $t \geq 0$, such that $x(t) \in \mathbb{R}^n$, $u(t), y(t) \in \mathbb{R}^m$, $(f_R(t), e_R(t)) \in \mathbb{R}^p \times \mathbb{R}^p$ is the resistive port which is terminated by a *resistive element* $\alpha : \mathbb{R}^p \to \mathbb{R}^p$. The continuous matrix-valued functions A, B_R, B are such that $A(x) \in \mathbb{R}^{n\times n}$, $A(x) + A^T(x) = 0$, $B_R(x) \in \mathbb{R}^{n\times p}$, $B(x) \in \mathbb{R}^{n\times m}$. The function $H \in \mathcal{C}^2(\mathbb{R}^n, \mathbb{R})$ is called the *Hamiltonian* of the system which often represents the energy. Port-controlled Hamiltonian systems have been extensively studied, see, for example, [35], [40], [58].

The following proposition gives a class of port-controlled Hamiltonian systems which belongs to the class of constant incremental passive systems. Recall that a function $f:\mathbb{R}^n\to\mathbb{R}^n$ is called *monotone and non-decreasing* if it satisfies

$$(f(a)-f(b))^T(a-b)\geq 0,$$

for any $a,b\in\mathbb{R}^n$ and it is called *strictly monotone increasing* if it satisfies

$$(f(a)-f(b))^T(a-b)>0,$$

for any $a\neq b$, $a,b\in\mathbb{R}^n$, see, for example, [50].

Proposition 5.5.1 *Suppose that the port-controlled Hamiltonian system* **P** *is described by* (5.24)–(5.27) *with a convex Hamiltonian H. Assume that the function α is monotone and non-decreasing, and the matrices A, B_R and B are constants.*

Then **P** *is constant incremental passive. More precisely, for any equilibrium point $(x_0,u_0)\in\mathbb{R}^n\times\mathbb{R}^m$ of* (5.24)–(5.27) *generating the output y_0, we define $H_{x_0}\in\mathcal{C}^2(\mathbb{R}^n,\mathbb{R}_+)$ by*

$$H_{x_0}(x)=H(x)-H(x_0)-\nabla H(x_0)(x-x_0). \tag{5.28}$$

Then **P** *is dissipative w.r.t. the storage function H_{x_0} and the supply rate*

$$s(y,u)=\langle y-y_0,u-u_0\rangle.$$

PROOF. Let (x_0,u_0) be the equilibrium point of **P** generating the output y_0.

It is easy to see that the system **P** takes the same form of affine nonlinear systems assumed in Theorem 5.4.2 by taking there $f(x)=A(\nabla H(x))^T-B_R\alpha\left(B_R^T(\nabla H(x))^T\right)$ and $g=B$. Hence, to show the dissipativity of **P** w.r.t. the storage function H_{x_0} and the supply rate $s(y,u)=\langle y-y_0,u-u_0\rangle$, we can check the incremental stability condition (5.19).

$$\begin{aligned}&[\nabla H(x)-\nabla H(x_0)][f(x)-f(x_0)]\\&=-[\nabla H(x)B_R-\nabla H(x_0)B_R]\left[\alpha\left(B_R^T(\nabla H(x))^T\right)-\alpha\left(B_R^T(\nabla H(x_0))^T\right)\right]\\&\leq 0,\end{aligned}$$

where the last inequality is due to the monotonicity of α.

Since the above result holds for any equilibrium point (x_0,u_0), the claim follows from Theorem 5.4.2. □

Remark 5.5.2 Proposition 5.5.1 is related to the Interconnection and Damping Assignment (IDA) technique proposed in [40]. More precisely, let the plant **P** be as in Proposition 5.5.1 with constant α and $m \leq n$. The objective of IDA controller (see also [40, Proposition 1]) is to assign a static feedback to **P** such that the closed-loop system has the form

$$\dot{x} = (A_d(x) - B_d(x))(\nabla H_d(x))^T, \tag{5.29}$$

where $A_d(x) = A + A_a(x)$ and $B_d(x) = B_R \alpha B_R^T + B_a(x)$ are desired interconnection and damping matrices, and $H_d(x) = H(x) + H_a(x)$ is the closed-loop storage function which has a minimum at a desirable equilibrium state x_0. The functions $A_a(x), B_a(x)$ and $H_a(x)$ are chosen by the controller designer. The IDA objective can be achieved using [40, Proposition 1] by taking $A_a(x) = 0$, $B_a(x) = 0$ and $H_a = -\nabla H(x_0)(x - x_0) - H(x_0)$; and the solution to the matching equation (see [40] for the definition)

$$-gu_0 = (A - B_R \alpha B_R^T)(\nabla H(x_0))^T,$$

is given by (5.18). Indeed, using the static feedback $u = u_0$, the closed-loop system has the port-controlled Hamiltonian structure as in (5.29) with the closed-loop storage function $H_d = H_{x_0}$. □

In [21], a large class of electrical circuits which satisfies the system in Proposition 5.5.1 is presented. It is also shown that these circuits are stabilizable by a PI controller. This class includes circuits using nonlinear inductors with strictly convex magnetic energy and nonlinear capacitors with strictly convex electric energy.

Another related work to the new storage function as in (5.28) is presented in Maschke *et al* [35]. In [35], a Lyapunov function is constructed to show the stability of a port-controlled Hamiltonian system forced by a constant input. The relation between these two functions can be drawn if we assume that the matrices $J(x), R(x)$ and $g(x)$ as in [35] are constants, and $(J - R)$ is invertible.

Consider the systems in [35] with constant $J(x), R(x)$ and $g(x)$,

$$\dot{x} = (J - R)(\nabla H(x))^T + gu,$$

where $J \in \mathbb{R}^{n \times n}$ is skew-symmetric matrix, $R \in \mathbb{R}^{n \times n}$ is nonnegative symmetric matrix and $g \in \mathbb{R}^{n \times m}$. Suppose that the system is forced by a constant input $u_0 \in \mathbb{R}^m$ and $(J - R)$ is invertible. If for some constant $x_0 \in \mathbb{R}^n$, the equation $(\nabla H(x_0))^T = (J - R)^{-1} g u_0$ is solvable then the point (x_0, u_0) is the equilibrium of the system. By direct application of [35, Theorem 6.1], we have the following Lyapunov function to describe the stability of x_0 for the forced system (note that we use the same notations H_r, K and c as in [35]),

$$H_r(x) = H(x) - u_0^T K^T x - cu_0,$$

where $K = (J-R)^{-1}g$ and $c \in \mathbb{R}^m$ is an arbitrary constant to ensure the positive definiteness of H_r. By using $(\nabla H(x_0))^T = Ku_0$ and by taking $c = (\nabla H(x_0)x_0 + H(x_0))\frac{u_0^T}{\|u_0\|^2}$, we have

$$H_r(x) = H(x) - \nabla H(x_0)(x-x_0) - H(x_0),$$

which is identical to the storage function H_{x_0} as in (5.28).

Example 5.5.3 Consider again the electrical circuit in Example 5.2.2. The system equations **P** in the port-controlled Hamiltonian formulation is given by

$$\begin{aligned}
\begin{bmatrix}\dot{\phi}\\ \dot{q}\end{bmatrix} &= \begin{bmatrix}0 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial\phi}H(\phi,q)\\ \frac{\partial}{\partial q}H(\phi,q)\end{bmatrix} + \begin{bmatrix}1\\ 0\end{bmatrix} f_R + \begin{bmatrix}0\\ 1\end{bmatrix} u,\\
\begin{bmatrix}e_R\\ y\end{bmatrix} &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}\frac{\partial}{\partial\phi}H(\phi,q)\\ \frac{\partial}{\partial q}H(\phi,q)\end{bmatrix}\\
f_R &= -\alpha(e_R),
\end{aligned}$$

where $z(t) = \begin{bmatrix}\phi(t)\\ q(t)\end{bmatrix} \in \mathbb{R}^2$ is the state, $I \in \mathbb{R}$ is the input current, $y \in \mathbb{R}$ is the output voltage, $(f_R(t), e_R(t))$ is the resistive port which is terminated by a strictly monotone increasing resistor $\alpha \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ satisfying $\alpha(0) = 0$. The state ϕ is the magnetic flux of the inductor, so that the voltage across the inductor is given by $V_L = \dot{\phi}$. The state q is the electric charge in the capacitor, so that the current flowing through the capacitor is given by $I_1 = \dot{q}$.

Here, the storage function is given by $H(\phi,q) = \int_0^\phi \bar{I_2}(\lambda)\mathrm{d}\lambda + \int_0^q \bar{V}(\lambda)\mathrm{d}\lambda$ where $\bar{I_2}, \bar{V} \in \mathcal{C}^1(\mathbb{R},\mathbb{R})$ are strictly monotone increasing, $\bar{I_2}(0) = 0$ and $\bar{V}(0) = 0$. Thus $\frac{\partial}{\partial\phi}H(\phi,q) = \bar{I_2}(\phi)$ represents the current flows through the inductor and $\frac{\partial}{\partial q}H(\phi,q) = \bar{V}(q)$ represents the voltage across the capacitor.

Note that for a linear resistor, $\alpha(e_R) = Re_R$, where $R > 0$ is the resistance, for a linear inductor, $\bar{I_2}(\phi) = \frac{1}{L}\phi$, where $L > 0$ is the inductance, and for a linear capacitor, $\bar{V}(q) = \frac{1}{C}q$, where $C > 0$ is the capacitance.

It is easy to see that all the assumptions in Proposition 5.5.1 are satisfied. Hence for any equilibrium point $(z_0 = \begin{bmatrix}\phi_0\\ q_0\end{bmatrix}, I_0)$ generating the output y_0, the plant **P** is dissipative w.r.t. the new storage function

$$\begin{aligned}
H_{z_0}(\phi,q) &= \int_{\phi_0}^{\phi} \bar{I_2}(\lambda_1)\,\mathrm{d}\lambda_1 - \frac{\partial}{\partial\phi}H(\phi_0,q_0)(\phi-\phi_0)\\
&\quad + \int_{q_0}^{q} \bar{V}(\lambda_2)\,\mathrm{d}\lambda_2 - \frac{\partial}{\partial q}H(\phi_0,q_0)(q-q_0) \qquad (5.30)\\
& \qquad (5.31)
\end{aligned}$$

and the supply rate $\langle y - y_0, I - I_0 \rangle$.

Now, suppose that $\bar{I}_2$ and $\bar{V}$ are invertible functions so that there exists functions $\Phi : \mathbb{R} \to \mathbb{R}$ and $Q : \mathbb{R} \to \mathbb{R}$ such that $\Phi \circ \bar{I}_2 = Id = \bar{I}_2 \circ \Phi$ and $Q \circ \bar{V} = Id = \bar{V} \circ Q$ where Id denotes the identity operator. By denoting $I_{20} = \bar{I}_2(\phi_0)$, $I_2 = \bar{I}_2(\phi)$, $V_0 = \bar{V}(q_0)$ and $V = \bar{V}(q)$, it follows from (5.30) that

$$
\begin{aligned}
H_{z_0}(\Phi(I_2), Q(V)) &= \int_{I_{20}}^{I_2} \sigma_1 \, \mathrm{d}\Phi(\sigma_1) - [\Phi(I_2) - \Phi(I_{20})] I_{20} \\
&\quad + \int_{V_0}^{V} \sigma_2 \, \mathrm{d}Q(\sigma_2) - [Q(V) - Q(V_0)] V_0 \\
&= \Phi(I_2) I_2 - \Phi(I_{20}) I_{20} - \int_{I_{20}}^{I_2} \Phi(\sigma_1) \, \mathrm{d}\sigma_1 - [\Phi(I_2) - \Phi(I_{20})] I_{20} \\
&\quad + Q(V)V - Q(V_0)V_0 - \int_{V_0}^{V} Q(\sigma_2) \, \mathrm{d}\sigma_2 - [Q(V) - Q(V_0)] V_0 \\
&= \Phi(I_2)(I_2 - I_{20}) - \int_{I_{20}}^{I_2} \Phi(\sigma_1) \, \mathrm{d}\sigma_1 \qquad (5.32) \\
&\quad + Q(V)(V - V_0) - \int_{V_0}^{V} Q(\sigma_2) \, \mathrm{d}\sigma_2, \qquad (5.33)
\end{aligned}
$$

where the first equality is obtained by applying in (5.30) the coordinate transformation $\lambda_1 = \Phi(\sigma_1)$ and $\lambda_2 = Q(\sigma_2)$, and we use integration by parts to get the second equality. Thus the storage function H_{z_0} as in (5.33) is similar to the storage function H_{x_0} as in (5.11). This relation can be established due to the existence of invertible functions $\bar{I}_2$ and $\bar{V}$.

5.6 Interconnection of constant incremental passive systems

Consider the standard feedback interconnection of two systems $\mathbf{P}_1$ and $\mathbf{P}_2$ as in Figure 5.2. Assume that $\mathbf{P}_j$, $j = 1, 2$, is described by

$$
\mathbf{P}_j \quad : \quad \dot{x}_j = f_j(x_j, u_j), \qquad y_j = h_j(x_j, u_j), \qquad (5.34)
$$

where $x_j(t) \in \mathbb{R}^{n_j}$ and $u_j(t), y_j(t) \in \mathbb{R}^m$.

It is known that if both $\mathbf{P}_1$ and $\mathbf{P}_2$ are passive nonlinear systems then the closed-loop system is passive with input $\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ and output $\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$ (see, for example, [58, Chapter 3], [59]). The interconnection of two constant incremental passive systems in the standard feedback interconnection as in Figure 5.2 also gives a constant incremental passive closed-loop system.

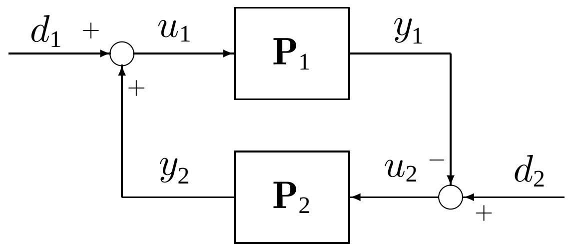


Figure 5.2: The feedback interconnection of systems $\mathbf{P}_1$ and $\mathbf{P}_2$.

Proposition 5.6.1 *Let $\mathbf{P}_j$, $j = 1,2$, given by (5.34) be constant incremental passive systems and consider the feedback interconnection as in Figure 5.2. Then the closed-loop system is constant incremental passive system.*

PROOF. Let $z = \left[\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right]$ be the state of the closed-loop system and $v = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$ be the input of the closed-loop system. Suppose that (z_0, v_0) be an equilibrium of the closed-loop system generating the output $\left[\begin{smallmatrix} y_{10} \\ y_{20} \end{smallmatrix}\right]$. By using $v_0 = \begin{bmatrix} d_{10} \\ d_{20} \end{bmatrix}$ and $z_0 = \left[\begin{smallmatrix} x_{10} \\ x_{20} \end{smallmatrix}\right]$, it is easy to see that $(x_{10}, u_{10} = d_{10} + y_{20})$ is an equilibrium of $\mathbf{P}_1$ and $(x_{20}, u_{20} = d_{20} - y_{10})$ is an equilibrium of $\mathbf{P}_2$. By the constant incremental passivity of $\mathbf{P}_j$, $j = 1,2$, there exists $H_{x_{j0}} \in \mathcal{C}^1(\mathbb{R}^{n_j}, \mathbb{R}_+)$ such that $\dot{H}_{x_{j0}}(x_j) \leq \langle y_j - y_{j0}, u_j - u_{j0} \rangle$ along the trajectory of $\mathbf{P}_j$. Using $H_{z_0}(z) = H_{x_{10}}(x_1) + H_{x_{20}}(x_2)$ and using the fact that $u_1 - u_{10} = d_1 - d_{10} + y_2 - y_{20}$ and $u_2 - u_{20} = d_2 - d_{20} - y_1 + y_{10}$, we obtain $\dot{H}_{z_0}(z)$ along the trajectories of the closed-loop system as follows

$$
\begin{aligned}
\dot{H}_{z_0}(z) &\leq \langle y_1 - y_{10}, u_1 - u_{10} \rangle + \langle y_2 - y_{20}, u_2 - u_{20} \rangle \\
&= \left\langle \begin{bmatrix} y_1 - y_{10} \\ y_2 - y_{20} \end{bmatrix}, \begin{bmatrix} d_1 - d_{10} \\ d_2 - d_{20} \end{bmatrix} \right\rangle.
\end{aligned}
$$

Thus, the closed-loop system is dissipative with respect to storage function H_{z_0} and supply rate $\left\langle \left[\begin{smallmatrix} y_1 - y_{10} \\ y_2 - y_{20} \end{smallmatrix}\right], \left[\begin{smallmatrix} d_1 - d_{10} \\ d_2 - d_{20} \end{smallmatrix}\right] \right\rangle$. For any equilibrium point (z_0, v_0), we can construct the storage function H_{z_0} as above. $\square$

5.7 Tracking of constant reference signal

Recall from Definition 5.1.3 that we denote by $\mathcal{R}$ the set of achievable constant reference for the plant $\mathbf{P}$ as in (2.3). Note that $0 \in \mathcal{R}$. Hence for any $y_0 \in \mathcal{R}$, there exists an equilibrium point (x_0, u_0) satisfying $h(x_0) = y_0$.

Let us consider the feedback system in Figure 5.3(a), where $y_0 \in \mathcal{R}$ and the disturbance d is generated by the exosystems $\mathbf{E}$ described by (4.1). The plant $\mathbf{P}$ is passive and is defined by (2.3) with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}^m$ and output $y(t) \in \mathbb{R}^m$. For the problem formulation given below, we consider the controller $\mathbf{C}$ as in (4.2)–(4.3) with state $x_c(t) \in \mathbb{R}^l$, input $e(t) \in \mathbb{R}^m$ and output $y_c(t) \in \mathbb{R}^m$.

Definition 5.7.1 In the closed-loop system **L** shown in Figure 5.3(a), the controller **C** solves the *output regulation for the plant* **P**, *the exosystem* **E** *and the set of achievable constant reference* $\mathcal{R}$ *locally*, if for any constant reference signal $y_0 \in \mathcal{R}$, there exists an open set $\mathcal{X} \times \mathcal{X}_c \times \mathcal{W} \subset \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^p$ (which depends on the equilibrium point (x_0, u_0) of **P** generating y_0) such that for any initial conditions $(x(0), x_c(0), w(0)) \in \mathcal{X} \times \mathcal{X}_c \times \mathcal{W}$, all state trajectories of the closed-loop system are bounded and $e(t) \to 0$ as $t \to \infty$. **C** solves *output regulation for* **P**, **E** *and* $\mathcal{R}$ *globally*, if for any constant reference signal $y_0 \in \mathcal{R}$ and for any initial conditions $(x(0), x_c(0), w(0)) \in \mathbb{R}^n \times \mathbb{R}^l \times \mathbb{R}^p$, all state trajectories of the closed-loop system are bounded and $e(t) \to 0$ as $t \to \infty$.

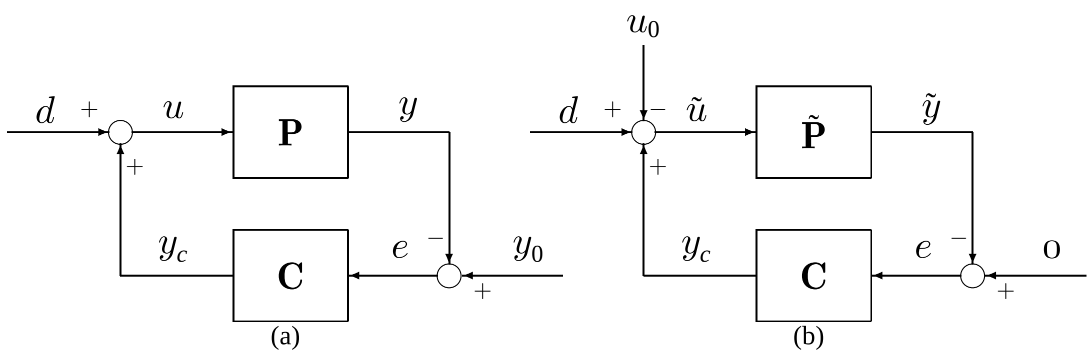


Figure 5.3: (a). The output regulation problem for the plant **P**, the disturbance d generated by an exosystem **E** and the constant reference signal $y_0 \in \mathcal{R}$ is to find a controller **C** such that the state trajectories of the closed-loop system **L** are bounded and $e(t) \to 0$ as $t \to \infty$. (b). If **P** is constant incremental passive system then the output regulation problem for **P**, **E** and $\mathcal{R}$ can be transformed into the disturbance rejection problem for a passive plant $\tilde{\mathbf{P}}$ with input $\tilde{u} = u - u_0$, output $\tilde{y} = y - y_0$ and an exosystem $\tilde{\mathbf{E}}$ generating disturbance $\tilde{d} = d - u_0$.

Suppose that the plant **P** is constant incremental passive then it is possible to transform the output regulation problem for **P**, **E** and $\mathcal{R}$ into the disturbance rejection problem for a passive plant $\tilde{\mathbf{P}}$ with input $\tilde{u} = u - u_0$, output $\tilde{y} = y - y_0$ and the disturbance $\tilde{d} = d - u_0$ generated by an extended exosystem $\tilde{\mathbf{E}}$ as shown in Figure 5.3(b). Thus we can apply the result of disturbance rejection from Chapter 4 to $\tilde{\mathbf{P}}$. For lucidity, we describe in the proposition below the case where the disturbance d is a constant signal and it will be generalized later to the case where d is generated by an exosystem.

Proposition 5.7.2 *Assume the plant* **P** *defined by* (2.3) *is constant incremental passive and for any equilibrium point* (x_0, u_0), **P** *has storage function* H_{x_0} *satisfying* (5.3), $H_{x_0}(x) > 0$ *for* $x \neq x_0$ *and* $H_{x_0}(x_0) = 0$. *Denote by* $\mathcal{R}$ *the set of achievable constant reference for* **P**.

Let the controller $\mathbf{C}$ *be given by*

$$\dot{x}_c = Be, \qquad y_c = B^T x + De, \tag{5.35}$$

where $x_c(t) \in \mathbb{R}^m$ *is the controller state,* $e(t) \in \mathbb{R}^m$ *is the controller input,* $y_c(t) \in \mathbb{R}^m$ *is the controller output, the matrix* $B \in \mathbb{R}^{m\times m}$ *is full rank and the matrix* $D \in \mathbb{R}^{m\times m}$ *satisfies* $D = D^T \geq kI$, $k > 0$. *Consider the control system* $\mathbf{L}$ *as in Figure 5.3(a) with a constant reference signal* $y_0 \in \mathcal{R}$ *and a constant disturbance* $d(t) = d_0 \in \mathbb{R}^m$ *for all* $t \geq 0$ *generated by an exosystem* $\mathbf{E}$.

(i) *Then* $\mathbf{C}$ *solves the output regulation problem for* $\mathbf{P}$, $\mathbf{E}$ *and* $\mathcal{R}$ *locally.*

(ii) *If for any equilibrium point* (x_0, u_0), H_{x_0} *is proper then* $\mathbf{C}$ *solves the output regulation problem for* $\mathbf{P}$, $\mathbf{E}$ *and* $\mathcal{R}$ *globally.*

(iii) *Moreover, let the equilibrium point* (x_0, u_0) *generating the output* y_0 *be unique and the system* $\tilde{\mathbf{P}}$ *defined by*

$$\begin{aligned} \dot{\tilde{x}} &= f(\tilde{x} + x_0, \tilde{u} + u_0), \\ \tilde{y} &= h(\tilde{x} + x_0) - y_0, \end{aligned} \tag{5.36}$$

is zero-state detectable where the state $\tilde{x}(t) \in \mathbb{R}^n$, *the input signal* $\tilde{u}(t) \in \mathbb{R}^m$, *the output* $\tilde{y}(t) \in \mathbb{R}^m$, *the mappings* f *and* h *are as in (2.3). Assume that there is no pole-zero cancellation between* $\tilde{\mathbf{P}}$ *and* $\mathbf{C}$ *(as in Definition 4.1.2). Then* $\lim_{t\to\infty} \left\| \begin{bmatrix} x(t)-x_0 \\ x_c(t)+(B^T)^{-1}(d_0-u_0) \end{bmatrix} \right\| \to 0.$

PROOF. Proof of **(i)**. Let $y_0 \in \mathcal{R}$ and suppose that (x_0, u_0) be an equilibrium point of $\mathbf{P}$ generating y_0. Since $B \in \mathbb{R}^{m\times m}$ is full rank, it is invertible. Consider the storage function $H_L(x, x_c) = H_{x_0}(x) + \frac{1}{2}\|x_c + (B^T)^{-1}(d_0 - u_0)\|^2$. By the assumption on H_{x_0}, the storage H_L has a global minimum at $\big(x_0, -(B^T)^{-1}(d_0 - u_0)\big)$ which is also the equilibrium point of the closed-loop state equation which includes the external constant signal y_0 and d_0. Using (5.3), (5.35) and the interconnections $u = y_c + d_0$, $e = y_0 - y$, we obtain that

$$\begin{aligned} \dot{H}_L &\leq \langle y - y_0, u - u_0\rangle + \langle x_c + (B^T)^{-1}(d_0 - u_0), Be\rangle \\ &= \big\langle y - y_0, B^T x_c - D(y - y_0) + d_0 - u_0\big\rangle - \langle B^T x_c + d_0 - u_0, y - y_0\rangle \\ &= -\langle y - y_0, D(y - y_0)\rangle \leq -k\|y - y_0\|^2. \end{aligned} \tag{5.37}$$

Using H_L as a Lyapunov function, it follows from (5.37) that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any initial conditions $\begin{bmatrix} x(0) \\ x_c(0) \end{bmatrix} \in \bar{\mathbf{B}}_\delta + \begin{bmatrix} x_0 \\ (B^T)^{-1}(u_0-d_0) \end{bmatrix}$ implies that $\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} \in \bar{\mathbf{B}}_\varepsilon + \begin{bmatrix} x_0 \\ (B^T)^{-1}(u_0-d_0) \end{bmatrix}$ for all $t \geq 0$. By choosing $d_0 \in \mathcal{W} = \mathbf{B}_{\delta/2}$, it follows that $(x(0), x_c(0)) \in \mathcal{X} \times \mathcal{X}_c = \mathbf{B}_{\delta/2} + \begin{bmatrix} x_0 \\ (B^T)^{-1}u_0 \end{bmatrix}$ implies $\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix} \in \bar{\mathbf{B}}_\varepsilon + \begin{bmatrix} x_0 \\ (B^T)^{-1}(u_0 - d_0) \end{bmatrix}$ for all $t \geq 0$, in particular, the state $\begin{bmatrix} x(t) \\ x_c(t) \end{bmatrix}$ is bounded.

From (5.37) we conclude that $e = y_0 - y$ is square integrable. Using the continuity of f, the boundedness of x implies that $\dot{x}$ is bounded, hence x is uniformly continuous. From the continuity of h we conclude that e is uniformly continuous. Hence $\lim_{t\to\infty} \|e(t)\| = 0$.

Proof of **(ii)**. If H_{x_0} is proper then H_L as in the proof of **(i)** is also proper. For any $(x(0), x_c(0)) \in \mathbb{R}^n \times \mathbb{R}^m$ and for any $d_0 \in \mathbb{R}^m$, by integrating (5.37) from 0 to t we obtain that $H_L(x(t), x_c(t)) \leq H_L(x(0), x_c(0))$ for all $t \geq 0$. By the properness of H_L, every state trajectory (x, x_c) remains bounded. Thus, we can apply the preceding argument to conclude that $\lim_{t\to\infty} e(t) = 0$.

Proof of **(iii)**. The plant $\mathbf{P}$ as in (2.3) with output $\tilde{y} = h(x) - y_0$ is equivalent to the plant $\tilde{\mathbf{P}}$ whose input is disturbed by a constant disturbance $-u_0$ (see Figure 5.3(b)). Indeed, if we substitute $\tilde{u} = u - u_0$ into (5.36) and use coordinate changes $x = \tilde{x} + x_0$, we get $\mathbf{P}$ as in (2.3) with output $\tilde{y} = y - y_0$. Hence the output regulation problem for $\mathbf{P}$, $\mathbf{E}$ and $\mathcal{R}$ is equivalent to the disturbance rejection problem for a passive plant $\tilde{\mathbf{P}}$ and an exosystem $\tilde{\mathbf{E}}$ generating constant disturbance $d_0 - u_0$.

By denoting $\tilde{x} = x - x_0$, it can be checked that the storage function $\tilde{H} : \tilde{x} \mapsto H_{x_0}(\tilde{x} + x_0)$ defines the passivity of $\tilde{\mathbf{P}}$, i.e., $\dot{\tilde{H}} \leq \langle \tilde{y}, \tilde{u} \rangle$. The storage function $\tilde{H}$ satisfies $\tilde{H}(\tilde{x}) > 0$ for $\tilde{x} \neq 0$ and $\tilde{H}(0) = 0$.

Using Lemma 4.1.3, the closed loop system $\tilde{\mathbf{L}}$ as in Figure 5.3(b) is zero-state detectable. Using the storage function $\tilde{H}$ and the Theorem 4.4.3, the controller $\mathbf{C}$ solves the disturbance rejection problem for $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{E}}$ locally. This implies that there exists $\delta > 0$ such that $\begin{bmatrix} \tilde{x}(0) \\ x_c(0) + (B^T)^{-1}(d_0 - u_0) \end{bmatrix} \in \bar{\mathbf{B}}_\delta \Rightarrow \begin{bmatrix} \tilde{x}(t) \\ x_c(t) + (B^T)^{-1}(d_0 - u_0) \end{bmatrix} \to 0$ as $t \to \infty$ and $\tilde{y}(t) \to 0$ as $t \to \infty$. In other words, for the original plant $\mathbf{P}$, there exists $\delta > 0$ such that $\begin{bmatrix} x(0) - x_0 \\ x_c(0) + (B^T)^{-1}(d_0 - u_0) \end{bmatrix} \in \bar{\mathbf{B}}_\delta \Rightarrow \begin{bmatrix} x(t) - x_0 \\ x_c(t) + (B^T)^{-1}(d_0 - u_0) \end{bmatrix} \to 0$ as $t \to \infty$ and $e(t) \to 0$ as $t \to \infty$. The same arguments can be carried out for any $y_0 \in \mathcal{R}$.

If H_{x_0} is proper then $\tilde{H}$ is also proper. Thus, using the same argument as before, the controller $\mathbf{C}$ solves the disturbance rejection problem for $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{E}}$ globally. This implies that the above result applies with $\delta = \infty$. $\square$

Proposition 5.7.2 shows that for constant incremental passive systems, we can solve the output regulation problem with constant reference signal by using a simple PI controller.

Let us examine again the plant $\mathbf{P}$ in Example 5.2.1 which is a simple integrator with a saturated output and $\mathcal{R} = (-1, 1)$. Consider the control block in Fig. 4.1 with $y_0 \in \mathcal{R}$ be a constant reference signal. It can be evaluated that by using a proportional gain feedback $y_c = Ke$, where $K > 0$ and $e = y_0 - y$, the tracking objective can be achieved whenever $d = 0$. However, if d is a non-zero constant disturbance, the closed loop system is only locally Input-to-State Stable (ISS, for definition see also [29],[56]), $e(t) \nrightarrow 0$ as $t \to \infty$ and the state can grow unboundedly for large constant disturbance, e.g., $|d| > 2K$.

It has been shown in Example 5.2.1 that **P** is a constant incremental passive system and satisfies all the assumption in Proposition 5.7.2. Using Proposition 5.7.2, a PI controller can be used to globally track constant reference signal $y_0 \in \mathcal{R}$ and to globally reject constant disturbance $d \in \mathbb{R}$.

It is easy to see that the electrical circuit in Example 5.2.2 with $\mathcal{R} = \mathbb{R}$ satisfies all the assumptions in Proposition 5.7.2. Therefore, we can use a PI controller for the electrical circuit to track any constant reference signal $y_0 \in \mathcal{R}$ and to reject any constant disturbance signal $d \in \mathbb{R}$.

The mechanical systems described in Example 5.2.3 is constant incremental passive systems with $\mathcal{R} = \{0\}$. In this example, PI controller can still be used to solve the output regulation problem (with the generalized velocity $\dot{q}$ as the regulated output). However, since it does not satisfy the detectability condition in Proposition 5.7.2**(iii)**, the convergence of $\begin{bmatrix} q(t) \\ \dot{q}(t) \end{bmatrix}$ to the desirable equilibrium $\begin{bmatrix} q_0 \\ 0 \end{bmatrix}$ cannot be obtained.

The technique used in Proposition 5.7.2 which transforms the output regulation problem into disturbance rejection problem allows us to deal with the general case where the disturbance d is not only a constant signal but generated by an exosystem **E** as in (4.1).

Theorem 5.7.3 *Assume the plant* **P** *defined by* (2.3) *is constant incremental passive and for any equilibrium point* (x_0, u_0), **P** *has storage function* H_{x_0} *satisfying* (5.3), $H_{x_0}(x) > 0$ *for* $x \neq x_0$ *and* $H_{x_0}(x_0) = 0$. *Denote by* $\mathcal{R}$ *the set of achievable constant reference for* **P**. *Let* **E** *be the exosystem from* (4.1) *generating the disturbance* d *with* $0 \in \sigma(S)$ *and denote by* χ *the minimal polynomial of* S. *Let the controller* **C** *be given by* (4.29) – (4.31) *where* S_{min} *has the characteristic polynomial* χ *and satisfies* $S_{min} + S_{min}^T = 0$. *Consider the control system* **L** *as in Figure* 5.3*(a) with constant reference signal* $y_0 \in \mathcal{R}$.

(i) *Then* **C** *solves the output regulation problem for* **P**, **E** *and* $\mathcal{R}$ *locally.*

(ii) *If for any equilibrium point* (x_0, u_0), H_{x_0} *is proper then* **C** *solves the output regulation problem for* **P**, **E** *and* $\mathcal{R}$ *globally.*

(iii) *Moreover, let the equilibrium point* (x_0, u_0) *generating the output* y_0 *be unique and the system* $\tilde{\mathbf{P}}$ *defined by*

$$\begin{array}{rcl} \dot{\tilde{x}} & = & f(\tilde{x}+x_0, \tilde{u}+u_0), \\ \tilde{y} & = & h(\tilde{x}+x_0)-y_0, \end{array} \tag{5.38}$$

is zero-state detectable where the state $\tilde{x}(t) \in \mathbb{R}^n$*, the input signal* $\tilde{u}(t) \in \mathbb{R}^m$*, the output* $\tilde{y}(t) \in \mathbb{R}^m$*, the mappings* f *and* h *are as in* (2.3)*. Assume that there is no pole-zero cancellation between* $\tilde{\mathbf{P}}$ *and* $\mathbf{C}$ *(as in Definition 4.1.2). Then* $\lim_{t\to\infty} \|x(t)-x_0\| \to 0$, $\lim_{t\to\infty} \|u(t)-u_0\| = 0$, $\lim_{t\to\infty} \dot{u}(t) = 0$ *and* $\lim_{t\to\infty} \|\dot{e}(t)\| = 0$.

PROOF. Proof of **(i)**. Let $y_0 \in \mathcal{R}$ and (x_0, u_0) be an equilibrium point of $\mathbf{P}$ generating y_0. Let the extended exosystem $\tilde{\mathbf{E}}$ generating the extended disturbance $\tilde{d}(t) = d(t) - u_0$ be given by

$$\begin{array}{rcl} \dot{\tilde{w}} & = & \tilde{S}\tilde{w}, \\ \tilde{d}(t) & = & \tilde{C}_w \tilde{w}, \end{array} \tag{5.39}$$

where $\tilde{w}(t) = \begin{bmatrix} w(t) \\ w_{yo}(t) \end{bmatrix}$, $w(t) \in \mathbb{R}^p$, $w_{yo}(t) \in \mathbb{R}^m$, $\tilde{C}_w = \begin{bmatrix} C_w & I^{m\times m} \end{bmatrix}$,

$$\tilde{S} = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}, \tag{5.40}$$

$C_w \in \mathbb{R}^{m\times p}$, $S \in \mathbb{R}^{p\times p}$ are as in (4.1) and $w_{yo}(0) = -u_0$. Note that since we assume $0 \in \sigma(S)$, the minimal polynomial of $\tilde{S}$ is also given by χ.

Let $\rho = x_c - \Sigma\tilde{w}$ where $\Sigma = -\phi_c^{-1}\phi_w$, ϕ_c is as in (4.32) and

$$\phi_w = \begin{bmatrix} (C_w \;\; I) \\ (C_w \;\; I)\tilde{S} \\ \vdots \\ (C_w \;\; I)\tilde{S}^{q-1} \end{bmatrix}.$$

Note that $B^T\Sigma + \tilde{C}_w = 0$ and $\Sigma\tilde{S} = A\Sigma$ as shown in the proof of Theorem 4.4.3.

It can be checked that the extended closed-loop system $\tilde{\mathbf{L}}$ which includes the exosystem can be described by the following state equation:

$$\dot{x} = f(x, B^T\rho + D(y_0 - h(x)) + u_0) \tag{5.41}$$

$$\dot{\rho} = A\rho + B(y_0 - h(x)) \tag{5.42}$$

$$y = h(x),$$

where we have used the interconnection $e = y_0 - h(x)$ and $u = y_c + d = B^T\rho + De + u_0$. Note that the point $\begin{bmatrix} x_0 \\ 0 \end{bmatrix}$ is an equilibrium point of $\tilde{\mathbf{L}}$.

Consider the storage function $H_L(x,\rho) = H_{x_0}(x) + \frac{1}{2}\|\rho\|^2$. By the assumption on H_{x_0}, the storage H_L has a global minimum at $(x_0, 0)$. Using (5.3), (5.41), (5.42), we obtain that

$$\begin{aligned} \dot{H}_L &\leq \langle y-y_0, u-u_0\rangle + \langle \rho, A\rho + B(y_0-y)\rangle \\ &= \big\langle y-y_0, B^T\rho - D(y-y_0)\big\rangle - \langle B^T\rho, y-y_0\rangle \\ &= -\langle y-y_0, D(y-y_0)\rangle \leq -k\|y-y_0\|^2, \end{aligned} \tag{5.43}$$

where we have used the skew-symmetry of A.

Using H_L as a Lyapunov function, it follows from (5.43) that for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any initial conditions $\begin{bmatrix} x(0)-x_0 \\ \rho(0) \end{bmatrix} \in \bar{\mathbf{B}}_\delta$ implies that $\begin{bmatrix} x(t)-x_0 \\ \rho(t) \end{bmatrix} \in \bar{\mathbf{B}}_\varepsilon$ for all $t \geq 0$. Since $\tilde{w}$ is bounded and since $\rho = x_c - \Sigma\tilde{w}$, it follows that $\begin{bmatrix} x \\ x_c \end{bmatrix}$ is bounded.

From (5.43) we conclude that $e = y_0 - y$ is square integrable. Following the same argument as in the proof of Proposition 5.7.2**(i)**, we have that e is uniformly continuous. Hence $\lim\limits_{t\to\infty}\|e(t)\| = 0$.

Proof of **(ii)**. If H_{x_0} is proper then H_L as in the proof above is also proper. Let $\begin{bmatrix} x(0) \\ \rho(0) \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}^{\kappa m}$. By integrating (5.43) from 0 to t we obtain that $H_L(x(t),\rho(t)) \leq H_L(x(0),\rho(0))$ for all $t \geq 0$. By the properness of H_L, every state trajectory $(x(t),\rho(t))$ remains bounded. Thus, using the preceding argument, we conclude that for any initial conditions $\begin{bmatrix} x(0) \\ x_c(0) \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}^{\kappa m}$, the state trajectory $\begin{bmatrix} x \\ x_c \end{bmatrix}$ is bounded and $\lim\limits_{t\to\infty} e(t) = 0$.

Proof of **(iii)**. The proof is similar to that of Proposition 5.7.2**(iii)**. The output regulation problem for $\mathbf{P}$, $\mathbf{E}$ and $\mathcal{R}$ is equivalent to the disturbance rejection problem for a passive plant $\tilde{\mathbf{P}}$ and an exosystem $\tilde{\mathbf{E}}$ as in (5.39). By denoting $\tilde{x} = x - x_0$, $\tilde{u} = u - u_0$ and $\tilde{y} = y - y_0$, it can be evaluated that the storage function $\tilde{H} : \tilde{x} \mapsto H_{x_0}(\tilde{x} + x_0)$ defines the passivity of $\tilde{\mathbf{P}}$, i.e., $\dot{\tilde{H}} \leq \langle \tilde{y}, \tilde{u}\rangle$. The storage function $\tilde{H}$ satisfies $\tilde{H}(\tilde{x}) > 0$ for $\tilde{x} \neq 0$ and $\tilde{H}(0) = 0$.

Using Lemma 4.1.3 and using the hypothesis in **(iii)**, the closed loop system $\tilde{\mathbf{L}}$ as in Figure 5.3(b) is zero-state detectable. Using the storage function $\tilde{H}$ and the Theorem 4.4.3, the controller $\mathbf{C}$ solves the disturbance rejection problem for $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{E}}$ locally. This implies that there exists $\delta > 0$ such that $\begin{bmatrix} \tilde{x}(0) \\ x_c(0)-\Sigma\tilde{w}(0) \end{bmatrix} \in \bar{\mathbf{B}}_\delta \Rightarrow \begin{bmatrix} \tilde{x}(t) \\ x_c(t)-\Sigma\tilde{w}(t) \end{bmatrix} \to 0$ as $t \to \infty$ and $\tilde{y}(t) \to 0$ as $t \to \infty$. In other words, for the original plant $\mathbf{P}$, there exists $\delta > 0$ such that $\begin{bmatrix} x(0)-x_0 \\ x_c(0)-\Sigma\tilde{w}(0) \end{bmatrix} \in \bar{\mathbf{B}}_\delta \Rightarrow \begin{bmatrix} x(t)-x_0 \\ x_c(t)-\Sigma\tilde{w}(t) \end{bmatrix} \to 0$ as $t \to \infty$ and $e(t) \to 0$ as $t \to \infty$. The same arguments can be carried out for any $y_0 \in \mathcal{R}$.

Using the result of Proposition 4.2.2 for $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{E}}$, we conclude that $\tilde{u}(t) \to 0$, $\dot{\tilde{u}}(t) \to 0$ and $\dot{\tilde{y}}(t) \to 0$ as $t \to \infty$. This implies that $u(t) - u_0 \to 0$, $\dot{u}(t) \to 0$ and $\dot{e}(t) \to 0$ as $t \to \infty$.

If H_{x_0} is proper then $\tilde{H}$ is also proper. Thus, using the same argument as before, the controller **C** solves the disturbance rejection problem for $\tilde{\mathbf{P}}$ and $\tilde{\mathbf{E}}$ globally. This implies that the above result applies with $\delta = \infty$. $\square$

Note that if $0 \notin \sigma(S)$ is not assumed in the exosystem in Theorem 5.7.3, the result still holds by extending S_{min} in the controller **C** so that 0 is included in the eigenvalues of the extended S_{min}.

The result of Theorem 5.7.3 uses high-order controller to ensure that there exists mapping Σ which is exploited in the proof of the theorem. The existence of the mapping Σ follows from the result developed in Chapter 4. If a lower order controller is desirable then the design is more difficult since we need to guarantee the existence of a mapping Σ satisfying (4.7) for the extended exosystem $\tilde{\mathbf{E}}$ in the proof of the theorem.

5.8 Example in voltage regulation problem

Consider again the electrical circuit in the Example 5.2.2, where the voltage V on the nonlinear load **P** should track a constant reference voltage y_0 and the current input I can be decomposed into $I = I_c + I_d$. We can only control the current source I_c. The main current I_d comes from an external power supply, for example, from an AC/DC converter with power factor precompensation, as discussed in [28], and we treat I_d as a disturbance. The current source I_d has a DC component which is approximately equal to the desired current through **P**, but any small deviation of the DC component, as well as the AC components of I_d, should be compensated by controlling I_c. The AC components of I_d correspond to the fundamental frequency of the power grid and its harmonics.

Let us consider the control block for the voltage regulation problem as in Fig. 5.4. We assume that I_d is the disturbance signal which is generated by the exosystem **E** as in (4.1) with $0 \in \sigma(S)$. By denoting $\tilde{x} = x - x_0$, it is easy to verify that $\tilde{\mathbf{P}}$ described by

$$
\begin{aligned}
\dot{\tilde{x}}_1 &= \left(\frac{\mathrm{d}\phi}{\mathrm{d}\tilde{x}_1}\right)^{-1} \left(-\alpha(\tilde{x}_1 + \alpha^{-1}(y_0)) + \tilde{x}_2 + y_0\right), \\
\dot{\tilde{x}}_2 &= \left(\frac{\mathrm{d}q}{\mathrm{d}\tilde{x}_2}\right)^{-1} \left(-(\tilde{x}_1 + \alpha^{-1}(y_0)) + u_0 + \tilde{u}\right), \\
\tilde{y} &= \tilde{x}_2 = -e,
\end{aligned}
$$

is zero state-observable with input $\tilde{u}$ and output $\tilde{y}$, i.e., $\tilde{y} = 0$ and $\tilde{u} = 0 \Rightarrow \tilde{x} = 0$.

It can be checked that there is no pole-zero cancellation of $\tilde{\mathbf{P}}$ and the controller **C** as in Theorem 5.7.3. Thus, the controller **C** as in Theorem 5.7.3 can be used to control I_c, for solving output (voltage) regulation problem for **P**, **E** and $\mathbb{R}$ locally. If the controller

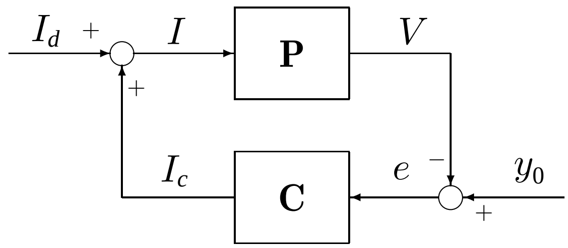


Figure 5.4: The voltage regulation control block where the objective of controller **C** is to reject the disturbance signal I_d, to track constant reference voltage y_0 and to keep the closed-loop state trajectories bounded.

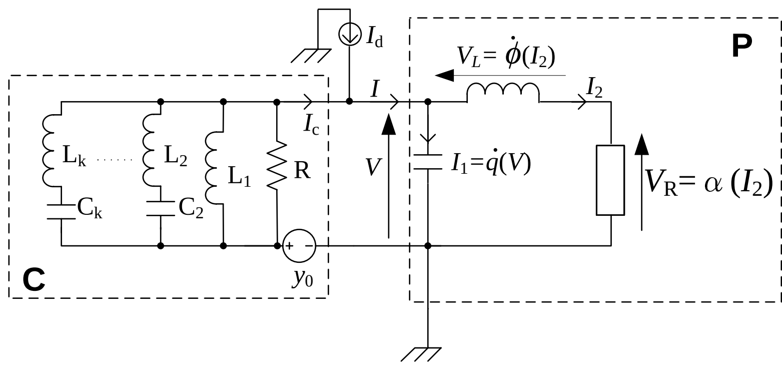


Figure 5.5: One realization of controller **C** for voltage regulation as in Fig. 5.4, where L_1 determines the gain of the integrator and the pair of inductor L_k and capacitor C_k, $k = 2, \ldots, \nu$, determine the frequencies of the signal to be rejected from the current I_d, i.e., $\omega_k = \sqrt{\frac{1}{L_k C_k}}$.

for I_d is able to produce the desired current $I_d(t) = \alpha^{-1}(y_0)$ for all $t \geq 0$, then $I_c(t) \to 0$ and $e(t) \to 0$ as $t \to \infty$. One such realization of the controller **C** is shown in Fig. 5.5.

Moreover, if ϕ and q is such that $\frac{\mathrm{d}\phi(I_2)}{\mathrm{d}I_2} \geq \varepsilon > 0$ and $\frac{\mathrm{d}q(V)}{\mathrm{d}V} \geq \varepsilon > 0$ for all $I_2 \in \mathbb{R}$ and $V \in \mathbb{R}$, then for any $y_0 \in \mathbb{R}$, H_{x_0} is proper. Thus, the same controller **C** solves the output regulation problem globally.

Since the controller we have discussed exploits the passivity of nonlinear plant and induces an output strictly passive closed-loop system, it can be expected that the closed-loop system possesses L_2-stability property with input $u - u_0$ and output $y - y_0$ and has some robustness property with respect to parameter uncertainties. Indeed, in the voltage regulation example above, for any additive parameter uncertainties in the inductance $\Delta\phi(I_2)$, in the capacitance $\Delta q(V)$ and/or in the resistance $\Delta\alpha(I_2)$ such that $(\phi + \Delta\phi)$, $(q + \Delta q)$ and/or $(\alpha + \Delta\alpha)$ are strictly monotone increasing, the same controller still assures tracking error e to converge to zero, while the plant state x and the controller state x_c converge to different values.

Figure 5.6 shows the simulation example where we consider $\phi : I_2 \mapsto \sqrt[3]{I_2}$, $q : V \mapsto \tanh(V)$ and $\alpha : I_2 \mapsto I_2 + (I_2)^3 + (I_2)^5$. The constant reference signal is $y_0 = 0.5$ and the exosystem **E** is as in (4.1) with

$$S = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & \Omega_1 & 0 & 0 \\ 0 & 0 & \Omega_2 & 0 \\ 0 & 0 & 0 & \Omega_3 \end{bmatrix}, \quad \Omega_k = \begin{bmatrix} 0 & k\pi 100 \\ -k\pi 100 & 0 \end{bmatrix}, \qquad k = 1,2,3,$$

$$C_w = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

The controller **C** is as in Theorem 5.7.3 with $A = S$, $B = \begin{bmatrix} 10 & 0 & 30 & 0 & 30 & 0 & 30 \end{bmatrix}$ and $D = -2$. For comparison, we use a proportional controller where $y_c = Ke$ with $K = 2$. For numerical simulations, the initial conditions is randomly chosen as follow

$$\begin{aligned} w(0) &= \begin{bmatrix} 0.643 & 0.289 & 0.636 & 0.320 & -0.316 & -0.421 & -0.318 \end{bmatrix}^T, \\ x_c(0) &= \begin{bmatrix} 0.068 & 0.454 & -0.381 & 0.677 & 0.136 & -0.259 & 0.405 \end{bmatrix}^T, \end{aligned}$$

and $\begin{bmatrix} I_2(0) \\ V(0) \end{bmatrix} = \begin{bmatrix} 0.4642 \\ -0.5493 \end{bmatrix}$.

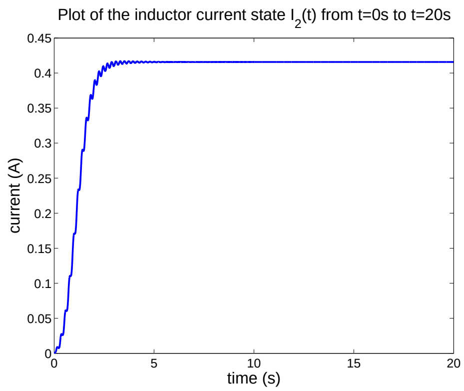
The simulation result as shown in Figure 5.6(a) and 5.6(b) demonstrates that the state trajectories $\left[\begin{smallmatrix} I_2 \\ V \end{smallmatrix}\right]$ are bounded and $V(t)$ converges to the desired constant reference $y_0 = 0.5$ as $t \to \infty$.

Figure 5.6(c) presents the plot of power spectral density of e using the proportional controller (the upper curve) and using the controller **C** as in Theorem 5.7.3. We computed the power spectral density of e using the windowed PSD algorithm from Matlab using the window from 960s to 1000s. From the plots in Figure 6.5(c), we see that the controller discussed in Theorem 5.7.3 is able to reject the disturbance d.

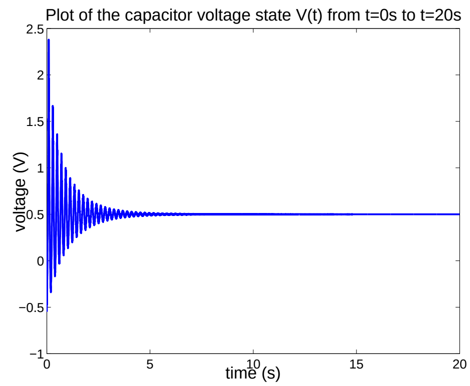
Suppose that the components in the load **P** has changed completely with $\phi : I_2 \mapsto \tanh(I_2)$, $q : V \mapsto \sqrt[3]{V}$ and $\alpha : I_2 \mapsto (I_2)^3$. The exosystems **E** and the constant reference y_0 remains the same as above. Using the same controller **C** as before, the simulation result as presented in Figure 5.7 shows that **C** is still able to solve the voltage regulation problem as long as the load **P** satisfies the constant incremental passivity property.

5.9 L^2 signal is added to the input disturbance

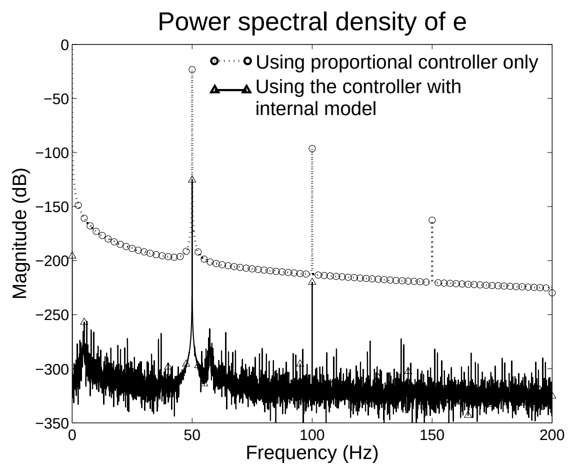
Combining the result from Section 5.7 and the result from Section 4.3, it is a fairly straightforward task to solve the problem of output regulation for a constant reference signal and with input disturbance which can be decomposed into an L^2 component and a component generated by an exosystem.



(a)

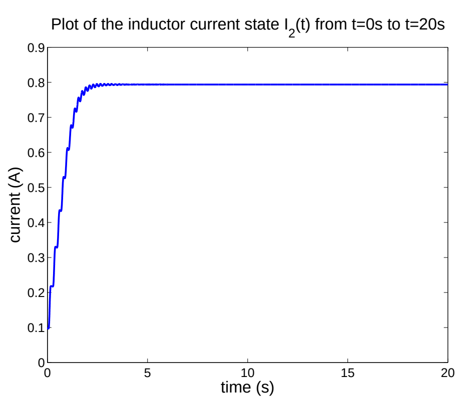


(b)

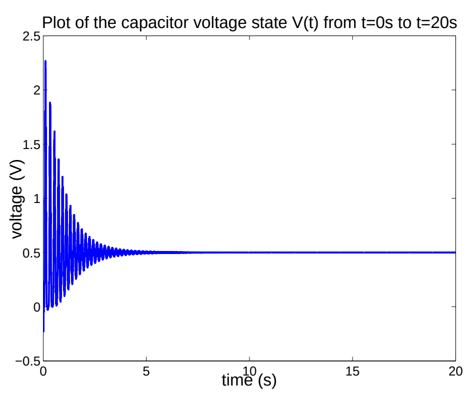


(c)

Figure 5.6: (a) The plot of the inductor current I_2 using the controller **C** from Theorem 5.7.3; (b) The plot of the capacitor voltage V using the controller **C** from Theorem 5.7.3; (c) The power spectral density of the tracking error e using the proportional controller with $y_c = Ke$ (upper curve) and using the controller **C** from Theorem 5.7.3 (lower curve) computed for the time interval $[960, 1000]$s.



(a)



(b)

Figure 5.7: The simulation result with different functions ϕ, q and α (a) The plot of the inductor current I_2 using the controller **C** from Theorem 5.7.3; (b) The plot of the capacitor voltage V using the controller **C** from Theorem 5.7.3;

Corollary 5.9.1 *Assume the plant* $\mathbf{P}$ *defined by* (2.3) *is constant incremental passive and assume* **(B1)-(B2)** *(as defined in Section 4.3). Suppose that for any equilibrium point* (x_0, u_0), $\mathbf{P}$ *has a proper storage function* H_{x_0} *satisfying* (5.3), $H_{x_0}(x) > 0$ *for* $x \neq x_0$ *and* $H_{x_0}(x_0) = 0$*. Denote by* $\mathcal{R}$ *the set of achievable constant reference for* $\mathbf{P}$*. Consider the disturbance* $d = d_0 + d_E$*, where* $d_0 \in L^2(\mathbb{R}_+, \mathbb{R}^m)$ *and* d_E *is generated by* $\mathbf{E}$ *as in* (4.1) *with* $0 \in \sigma(S)$ *and denote by* χ *the minimal polynomial of* S*. Let the controller* $\mathbf{C}$ *be given by* (4.29) – (4.31) *where* S_{min} *has the characteristic polynomial* χ *and satisfies* $S_{min} + S_{min}^T = 0$*. Consider the control system* $\mathbf{L}$ *as in Figure 5.3(a) with constant reference signal* $y_0 \in \mathcal{R}$*.*

Suppose that any equilibrium point (x_0, u_0) *generating the output* y_0 *is unique and for any* (x_0, u_0) *the system* $\tilde{\mathbf{P}}$ *as in* (5.36) *is zero-state detectable where the state* $\tilde{x}(t) \in \mathbb{R}^n$*, the input signal* $\tilde{u}(t) \in \mathbb{R}^m$*, the output* $\tilde{y}(t) \in \mathbb{R}^m$*, the mappings* f *and* h *are as in* (2.3)*. Assume that there is no pole-zero cancellation between* $\tilde{\mathbf{P}}$ *and* $\mathbf{C}$*.*

Then for any initial conditions $\begin{bmatrix} x(0) \\ x_c(0) \\ w(0) \end{bmatrix} \in \mathbb{R}^n \times \mathbb{R}^{\kappa m} \times \mathbb{R}^p$*, for any* $y_0 \in \mathcal{R}$ *and for any* $d_0 \in L^2(\mathbb{R}_+, \mathbb{R}^m)$*, the state trajectory* $\left[\begin{smallmatrix} x \\ x_c \end{smallmatrix}\right]$ *of* $\mathbf{L}$ *is bounded (hence there is no finite-escape time) and* $\|x(t) - x_0\| \to 0$ *as* $t \to \infty$ *(and hence* $e(t) \to 0$ *as* $t \to \infty$*).*

5.10 Incremental passivity and tracking of non-constant signals

The technique in dealing with the tracking of constant reference signals can be used to solve the tracking problem for a larger class of reference signals by generalizing the notion of constant incremental passivity. In Pavlov and Marconi [42], a notion of incremental passivity is introduced which extends the definition of constant incremental passivity.

Definition 5.10.1 The plant $\mathbf{P}$ as in (2.3) is *incremental passive* if for all $T \geq 0$, for any two solutions $x_i : [0, T] \to \mathbb{R}$ with input $u_i : [0, T] \to \mathbb{R}$ on $[0, T]$ where $i = 1, 2$ there exists a storage function $H \in \mathcal{C}^1(\mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}_+)$ such that

$$\dot{H}(t, x_1(t), x_2(t)) \leq \langle y_1(t) - y_2(t), u_1(t) - u_2(t) \rangle, \tag{5.44}$$

for all $t \in [0, T]$ where $y_i = h(x_i)$, $i = 1, 2$.

It can be checked that an LTI system is incremental passive. For nonlinear systems, it is an open problem how to characterize whether a given passive nonlinear systems is incremental passive and how to determine the corresponding storage function H.

However, if a given nonlinear systems is incremental passive then one can use the technique used in the tracking of constant reference signal to solve output regulation problem for non-constant reference signals.

Proposition 5.10.2 *Let the plant* $\mathbf{P}$ *defined by* (2.3) *be incremental passive such that it has a storage function* H *satisfying* (5.44) *and for all* $T \in \mathbb{R}_+$, $a \in \mathbb{R}^n$, *the function* $H(T, \cdot, a)$ *is proper. Suppose that* $y_r : \mathbb{R}_+ \to \mathbb{R}^m$ *is an admissible reference signal such that it can be generated by the state equation*

$$\dot{x}_r(t) = f(x_r(t), u_r(t)), \qquad y_r(t) = h(x_r(t)),$$

for all $t \in \mathbb{R}_+$, *where the functions* f *and* h *are as in* (2.3) *and the functions* $x_r : \mathbb{R}_+ \to \mathbb{R}^n$ *and* $u_r : \mathbb{R}_+ \to \mathbb{R}^m$ *are bounded.*

Assume that u_r *can be generated by an exosystem* $\mathbf{E}$ *from* (4.1) *with* $u_r = C_w w$ *and denote by* χ *the minimal polynomial of* S. *Let the controller* $\mathbf{C}$ *be given by* (4.29) – (4.31) *where* S_{min} *has the characteristic polynomial* χ *and satisfies* $S_{min} + S_{min}^T = 0$. *Consider the control system* $\mathbf{L}$ *as in Figure* 5.3*(a) with reference signal* y_r.

Then for any initial conditions $x(0) \in \mathbb{R}^n$ *the state trajectory* $\left[\begin{smallmatrix} x \\ x_c \end{smallmatrix}\right]$ *of* $\mathbf{L}$ *is bounded and* $\lim\limits_{t\to\infty} e(t) = 0$.

PROOF. Let $\rho = x_c + \Sigma w$ where $\Sigma = -\phi_c^{-1}\phi_w$, ϕ_c is as in (4.32) and ϕ_w is as in (4.34). Note that Σ satisfies $\Sigma S = A\Sigma$ and $B^T\Sigma + C_w = 0$. Define the storage function for $\mathbf{L}$ by $H_L(t,x,x_r,\rho) = H(t,x,x_r) + \frac{1}{2}\|\rho\|^2$. Using the incremental passivity of $\mathbf{P}$,

$$\begin{aligned}
\dot{H}_L(t,x(t),x_r(t),\rho(t)) &\le \langle y(t) - y_r(t), u(t) - u_r(t)\rangle + \langle \rho(t), A\rho(t) - B(y(t) - y_r(t))\rangle \\
&= \left\langle y(t) - y_r(t), B^T\rho(t) - D(y(t) - y_r(t))\right\rangle \\
&\quad - \langle \rho(t), B(y(t) - y_r(t))\rangle \\
&\le k\|y(t) - y_r(t)\|^2, \qquad (5.45)
\end{aligned}$$

where we have used $u(t) - u_r(t) = B^T\rho - (B^T\Sigma + C_w)w(t)$ in the second equation. The above inequality shows that for any $T \ge 0$,

$$H_L(T, x(T), x_r(T), \rho(T)) \le H_L(0, x(0), x_r(0), \rho(0)).$$

Since the function $H_L(T, \cdot, a, \cdot)$ is proper for any $T \ge 0$ and for any $a \in \mathbb{R}^n$, it follows from the above inequality that the state trajectory $\left[\begin{smallmatrix} x \\ \rho \end{smallmatrix}\right]$ is bounded. This implies that the controller state trajectory x_c is also bounded by the boundedness of the exosystems state w.

From (5.45) we conclude that $e = y_r - y$ is square integrable. Using the continuity of f, this also implies that $\dot{x}$ and $\dot{x}_r$ are bounded, hence x and x_r are uniformly continuous. From the continuity of h we also have that e is uniformly continuous, and we conclude $\lim_{t\to\infty} \|e(t)\| = 0$. $\square$

Chapter 6

Tracking and disturbance rejection for fully actuated mechanical systems

It has been shown in Chapter 4 that a simple LTI internal model can be used to solve the disturbance rejection problem for passive nonlinear plants. The disturbance d is assumed to be of the form $d = d_0 + d_E$, where $d_0 \in L^2([0,\infty),\mathbb{R}^m)$, and d_E is generated by an LTI exosystem.

In this chapter, the plant is a fully actuated mechanical system with the vector of generalized coordinates denoted by q, which should track a $\mathcal{C}^2$ reference signal r. We combine an LTI controller as in Chapter 4 with a *Slotine-Li* type adaptive controller (see Slotine and Li [52]) for rejecting a disturbance signal $d = d_0 + d_E$ and for asymptotically tracking r. We assume that the signals r, $\dot{r}$ and $\ddot{r}$ are available to the controller, but the controller does not know the parameters of the plant.

Our construction can be modified to allow the same LTI compensator to be combined with other passivity-based tracking controllers, for example, the passivity-based adaptive tracking controller in Slotine and Li [53] or the adaptive tracking controller with adaptive friction compensator in Panteley *et al* [41].

In Scherpen and Ortega [48], it is shown that by using the *Slotine-Li* controller and by adding to it a high gain proportional block from the tracking error to the input, the L^2 gain from the disturbance to the tracking error can be made arbitrarily small. However, this approach does not assure that the error converges to zero for a disturbance which is not in L^2. For a recent survey on tracking controllers for fully actuated mechanical systems we refer to Sage *et al* [45].

Results related to those in this chapter have appeared in Bonivento *et al* [4]. The controller in [4] uses an adaptive internal model to find the frequencies of the disturbance. There are problems with the proofs of the main results in [4]. First, the proofs of [4, Propositions 1 and 2] use the La Salle invariance principle, but this is not applicable to time-varying systems. When the inertia and Coriolis matrices depend on the generalized coordinates, then the closed-loop system obtained after using a *Slotine-Li* type controller will depend on the reference $r(t)$, and hence it is time-varying (see (6.13)–(6.15) below). Second, the time derivative of the Lyapunov function in the proofs of [4, Propositions 1 and 2] is not correct since it does not take into account the dependence of the inertia matrix to the time-varying reference signal. Third, in the proof of [4, Proposition 2], it is claimed that $\tilde{q}(t) \to 0$ as $t \to \infty$ implies that $\tilde{q}(t)$ converges to a constant vector which is not correct.

One major difference between our approach to the one in [4] is that we use an adaptive controller to deal with uncertainty in the plant parameter. Also we show that a unique solution exist for the closed-loop system for any $d_0 \in L^2$.

6.1 Main result

Consider the problem of tracking a $\mathcal{C}^2$ reference signal r for a fully actuated mechanical system with the generalized coordinates q, without precise knowledge of the plant parameters. It is known that in the absence of disturbances, the *Slotine-Li* adaptive controller from [52] achieves asymptotic tracking of r with bounded state trajectories. In this section, we show that the *Slotine-Li* static feedback law applied to a fully actuated mechanical system produces a time-varying passive system. Then, we combine the *Slotine-Li* adaptive controller with a linear controller to achieve asymptotic tracking of r and at the same time the rejection of any input disturbance that can be decomposed into an L^2 component and a component generated by an autonomous exosystem.

We consider a plant **P** described by the second-order differential equation

$$\mathcal{M}(q)\ddot{q} + \mathcal{D}(q,\dot{q})\dot{q} + g(q) = u, \tag{6.1}$$

which we call a *fully actuated mechanical system*. Such systems often originate from Euler-Lagrange equations for mechanical systems and they have been extensively studied, see Astolfi *et al* [3], Ortega *et al* [39], Koivo [30]. Here, $q(t) \in \mathbb{R}^n$ is the vector of *generalized coordinates*, $\mathcal{M}(q)$ is self-adjoint and

$$m_1 I \leq \mathcal{M}(q) \leq m_2 I, \qquad \text{where} \quad m_1, m_2 > 0, \tag{6.2}$$

$g(q)$ is a locally Lipschitz continuous function (which usually represents forces due to the potential energy) and $u(t) \in \mathbb{R}^n$ is the input (usually, forces or torques). The function $\mathcal{M}(\cdot)$ is assumed to be continuously differentiable and $\mathcal{D}(\cdot,\cdot)$ is assumed to be locally Lipschitz continuous. As usual, we denote $\dot{\mathcal{M}}(q,\dot{q}) = \sum_{j=1}^{n} \frac{\partial \mathcal{M}}{\partial q_j}\dot{q}_j$. The state of this system is the vector $\left[\begin{smallmatrix} q \\ \dot{q} \end{smallmatrix}\right]$. We assume that $J(q,\dot{q}) = \dot{\mathcal{M}}(q,\dot{q}) - 2\mathcal{D}(q,\dot{q})$ satisfies $J^T(q,\dot{q}) + J(q,\dot{q}) \leq 0$, so that

$$\left\langle \left(\frac{1}{2}\dot{\mathcal{M}} - \mathcal{D}\right) a, a \right\rangle \leq 0 \quad \forall a \in \mathbb{R}^n. \tag{6.3}$$

We remark that if $g(q) = (\nabla V(q))^T$, where $V \in \mathcal{C}^1(\mathbb{R}^n, \mathbb{R}_+)$ is called the *potential energy*, then the plant **P** with output signal $\dot{q}$ is *passive* with respect to the storage function $H(q,\dot{q}) = \frac{1}{2}\langle \mathcal{M}(q)\dot{q}, \dot{q}\rangle + V(q)$, i.e., $\dot{H} \leq \langle \dot{q}, u\rangle$. We mention that if $J^T + J = 0$ then this system is *energy preserving*, meaning that $\dot{H} = \langle \dot{q}, u\rangle$.

We assume that $r \in \mathcal{C}^2(\mathbb{R}_+, \mathbb{R}^n)$ and the signals $r, \dot{r}, \ddot{r}$ are available to the controller. The input signal u is the sum of a disturbance signal d and the control input z (generated by the controller that we shall design), see Figure 6.2(a).

We assume that $\mathcal{M}, \mathcal{D}$ and g are not known exactly, but we can express them in terms of unknown real parameters $\theta_1, \theta_2, \ldots \theta_m$ as follows:

$$\begin{aligned} \mathcal{M}(q) &= \sum_{i=1}^{m} \mathcal{M}_i(q)\theta_i + \mathcal{M}_0(q), \\ \mathcal{D}(q,\dot{q}) &= \sum_{i=1}^{m} \mathcal{D}_i(q,\dot{q})\theta_i + \mathcal{D}_0(q,\dot{q}), \\ g(q) &= \sum_{i=1}^{m} g_i(q)\theta_i + g_0(q), \end{aligned} \tag{6.4}$$

where $\mathcal{M}_i$ is of class $\mathcal{C}^1$ and $\mathcal{D}_i, g_i$ are locally Lipschitz continuous. For any $q, q_1, a, b \in \mathbb{R}^n$, we introduce the matrix $\Phi(q, q_1, a, b) \in \mathbb{R}^{n\times m}$ such that

$$\Phi(q,q_1,a,b)\theta = \left(\sum_{i=1}^{m} \mathcal{M}_i(q)\theta_i\right) a + \left(\sum_{i=1}^{m} \mathcal{D}_i(q,q_1)\theta_i\right) b + \sum_{i=1}^{m} g_i(q)\theta_i, \tag{6.5}$$

where $\theta = \begin{bmatrix} \theta_1 & \theta_2 & \cdots & \theta_m \end{bmatrix}^T$ is the parameter vector.

We describe a first feedback loop which is based on the *Slotine-Li* controller and which eliminates r from the picture, so that the problem is reduced to the input disturbance rejection problem. We denote by $\hat{\mathcal{M}}(q), \hat{\mathcal{D}}(q,\dot{q})$ and $\hat{g}(q)$ the estimates of $\mathcal{M}(q), \mathcal{D}(q,\dot{q})$ and $g(q)$ corresponding to the estimate $\hat{\theta}$ of the unknown parameter vector θ. (This means that $\hat{\mathcal{M}}(q)$ is obtained from (6.4) by replacing θ with $\hat{\theta}$, and similarly for $\hat{\mathcal{D}}(q,\dot{q})$ and $\hat{g}(q)$.)

Consider the feedback law

$$u = \hat{\mathcal{M}}\dot{\xi} + \hat{\mathcal{D}}\xi + \hat{g} + v, \tag{6.6}$$

where

$$\xi := \dot{r} + \Lambda(r - q), \qquad \Lambda = \Lambda^T \geq \mu I > 0, \tag{6.7}$$

and v is the new input signal, containing d and any other components of the control input z (to be designed). The estimated parameters $\hat{\theta}$ evolve according to

$$\dot{\hat{\theta}} = -\lambda \Phi(q, \dot{q}, \dot{\xi}, \xi)^T \zeta, \tag{6.8}$$

where $\zeta = \dot{q} - \xi$ and $\lambda \in \mathbb{R}^{m \times m}, \lambda = \lambda^T > 0$, see Figure 6.1. Substituting (6.6) into (6.1) gives

$$\mathcal{M}(q)\dot{\zeta} + \mathcal{D}(q,\dot{q})\zeta = \left[\hat{\mathcal{M}}(q) - \mathcal{M}(q)\right]\dot{\xi} + \left[\hat{\mathcal{D}}(q,\dot{q}) - \mathcal{D}(q,\dot{q})\right]\xi + \hat{g}(q) - g(q) + v. \tag{6.9}$$

Introducing the estimation error $\tilde{\theta} = \hat{\theta} - \theta$, we have $\hat{\mathcal{M}}(q) - \mathcal{M}(q) = \sum_{i=1}^{m} \mathcal{M}_i(q)\tilde{\theta}$, and we have similar formulas for $\hat{\mathcal{D}}(q,\dot{q}) - \mathcal{D}(q,\dot{q})$ and $\hat{g}(q) - g(q)$. Now using (6.5), the formula (6.9) becomes

$$\mathcal{M}(q)\dot{\zeta} + \mathcal{D}(q,\dot{q})\zeta = \Phi(q, \dot{q}, \dot{\xi}, \xi)\tilde{\theta} + v. \tag{6.10}$$

From (6.8) it is clear that

$$\dot{\tilde{\theta}} = -\lambda \Phi(q, \dot{q}, \dot{\xi}, \xi)^T \zeta. \tag{6.11}$$

A simple computation shows that, denoting $e = r - q$,

$$-\dot{e} - \Lambda e = \zeta. \tag{6.12}$$

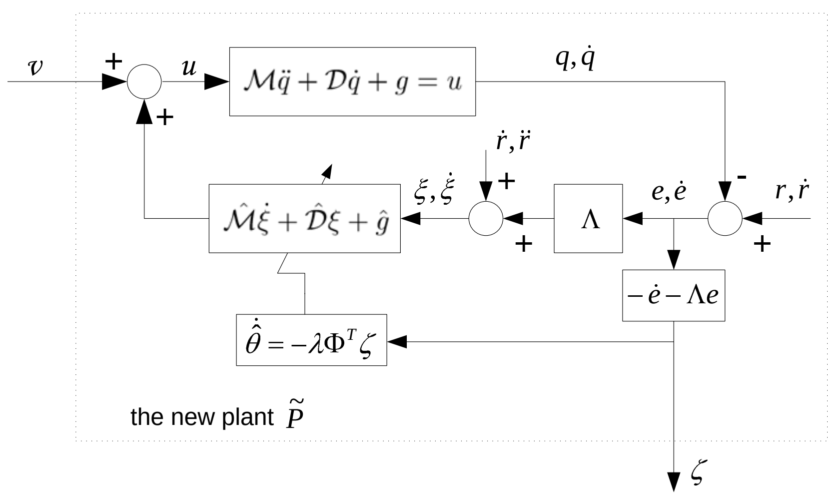


Figure 6.1: The new plant $\tilde{\mathbf{P}}$ obtained after the feedback (6.6), in which the signal r is internally generated. The tracking error is e. This is a time-varying passive system with input v, state $(e, \zeta, \tilde{\theta})$, and output ζ.

The differential equations (6.10), (6.11) and (6.12) determine a new system $\tilde{\mathbf{P}}$ (shown in Figure 6.1), for which it is natural to choose e, ζ and $\tilde{\theta}$ as state variables. What is disturbing in this system of equations is that (6.10), (6.11) depend also on

$q, \dot{q}$. However, the state variables q and $\dot{q}$ of the plant can be expressed in terms of the state of $\tilde{\mathbf{P}}$: $q = r - e$, $\dot{q} = \dot{r} + \zeta + \Lambda e$ (remember that r and $\dot{r}$ are regarded as known functions). Thus, it is possible to rewrite (6.10) and (6.11) without using $q, \dot{q}$:

$$\begin{aligned} \mathcal{M}_r(e,t)\dot{\zeta} + \mathcal{D}_r(e,\zeta,t)\zeta &= \Phi_r(e,\zeta,t)\tilde{\theta} + v, \quad (6.13)\\ \dot{\tilde{\theta}} &= -\lambda\Phi_r(e,\zeta,t)^T\zeta, \quad (6.14)\end{aligned}$$

where, by definition,

$$\mathcal{M}_r(e,t) = \mathcal{M}(r(t)-e), \qquad \mathcal{D}_r(e,\zeta,t) = \mathcal{D}(r(t)-e, \dot{r}(t)+\zeta+\Lambda e), \quad (6.15)$$

$$\Phi_r(e,\zeta,t) = \Phi\left(r(t)-e, \dot{r}(t)+\zeta+\Lambda e, \ddot{r}(t)-\Lambda(\Lambda e+\zeta), \dot{r}(t)+\Lambda e\right). \quad (6.16)$$

Thus, a neat description of $\tilde{\mathbf{P}}$ consists of (6.12), (6.13) and (6.14). We cannot know if this system of differential equations has a global solution for every initial state $(e(0), \zeta(0), \tilde{\theta}(0))$ and every input function $v \in L^2(\mathbb{R}_+, \mathbb{R}^n)$, and even if it does, we do not know if the solution is unique. We shall ignore these questions for the moment and we will show later that a unique global solution exists using our controller.

We denote

$$\dot{\mathcal{M}}_r = \frac{\partial \mathcal{M}_r}{\partial t} + \sum_{j=1}^{n} \frac{\partial \mathcal{M}_r}{\partial e_j}\dot{e}_j,$$

where $\frac{\partial \mathcal{M}_r(e)}{\partial t} = \sum_{j=1}^n \frac{\partial \mathcal{M}}{\partial q}(r-e)\dot{r}_j$. Note that $\dot{\mathcal{M}}_r(e,\zeta,t) = \dot{\mathcal{M}}(r(t)-e, \dot{r}(t)+\zeta+\Lambda e)$, so that (as in (6.3)),

$$\left\langle \left(\frac{1}{2}\dot{\mathcal{M}}_r - \mathcal{D}_r\right)a, a\right\rangle \le 0, \quad \forall a \in \mathbb{R}^n. \quad (6.17)$$

Using

$$\tilde{H}(e,\zeta,\tilde{\theta},t) = \frac{1}{2}\langle \mathcal{M}_r(e,t)\zeta, \zeta\rangle + \frac{1}{2}\langle \tilde{\theta}, \lambda^{-1}\tilde{\theta}\rangle \quad (6.18)$$

as a storage function, $\tilde{\mathbf{P}}$ is a time-varying passive system with input v, state $(e, \zeta, \tilde{\theta})$ and output ζ. Indeed, using (6.13) and (6.17), we have

$$\begin{aligned} \dot{\tilde{H}} &= \langle \mathcal{M}_r\zeta, \dot{\zeta}\rangle + \frac{1}{2}\langle \dot{\mathcal{M}}_r\zeta, \zeta\rangle + \langle \tilde{\theta}, -\Phi_r^T\zeta\rangle \\ &= -\langle \zeta, \mathcal{D}_r\zeta\rangle + \langle \zeta, \Phi_r\tilde{\theta} + v\rangle + \frac{1}{2}\langle \zeta, \dot{\mathcal{M}}_r\zeta\rangle - \langle \zeta, \Phi_r\tilde{\theta}\rangle \le \langle \zeta, v\rangle. \quad (6.19)\end{aligned}$$

Assume that a disturbance d acts on the original system in (6.1), meaning that it is added to the input u. This has the same effect as adding d to the input v of the new system $\tilde{\mathbf{P}}$. We connect a stabilizing controller $\mathbf{C}$ to $\tilde{\mathbf{P}}$, as shown in Figure 6.2(b). Thus, $v = y_c + d$ so that (according to (6.6)) $u = \hat{\mathcal{M}}\dot{\xi} + \hat{\mathcal{D}}\xi + \hat{g} + y_c + d$. Note that the total control input z from Figure 6.2(a) is $z = \hat{\mathcal{M}}\dot{\xi} + \hat{\mathcal{D}}\xi + \hat{g} + y_c$.

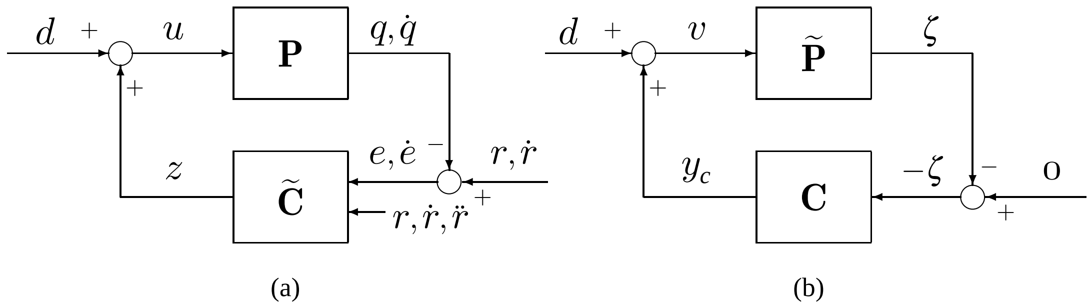


Figure 6.2: (a) The closed-loop system as in Theorem 6.1.5. Note that the controller $\tilde{\mathbf{C}}$ needs $r, \dot{r}$ and $\ddot{r}$. This block diagram is equivalent to the one shown in (b), where $\tilde{\mathbf{P}}$ is the new plant from Figure 6.1 and $\mathbf{C}$ is the stabilizing controller.

To put our result into perspective, first we restate the well-known result from Slotine and Li [52] and van der Schaft [58], which refers to the situation when $\mathbf{C}$ is a proportional feedback. In fact, our Proposition 6.1.3 below contains also a new part, which refers to the existence of global unique solution for any $d \in L^2(\mathbb{R}_+, \mathbb{R}^n)$ and the case when $r, \dot{r}$ and $\ddot{r}$ are bounded. The closed-loop system obtained from $\tilde{\mathbf{P}}$ with a proportional controller is shown in Figure 6.3, where $y_c = -D\zeta$, $D = D^T \geq kI$, $k > 0$. Recall that $e = r - q$.

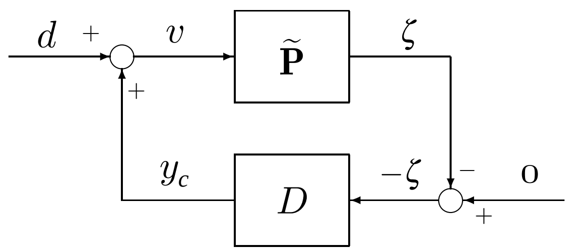


Figure 6.3: The new system $\tilde{\mathbf{P}}$ from Figure 6.1 with a proportional feedback $D = D^T \geq kI$, where $k > 0$ and a disturbance $d \in L^2(\mathbb{R}_+, \mathbb{R}^n)$. The result in van der Schaft [58] (the first part of our Proposition 6.1.3) states that $\zeta \in L^2(\mathbb{R}_+, \mathbb{R}^n)$, ζ and $\tilde{\theta}$ are bounded and $e(t) \to 0$ as $t \to \infty$. According to the second part of our Proposition 6.1.3, if $r, \dot{r}, \ddot{r}$ are bounded then $\zeta(t) \to 0$ and $\dot{e}(t) \to 0$.

We shall need the following slight generalization of Barbălat's lemma.

Lemma 6.1.1 *Suppose that $\zeta \in L^2(\mathbb{R}_+, \mathbb{R}^n)$ is uniformly continuous.*

Then $\lim\limits_{t \to \infty} \zeta(t) = 0$.

PROOF. Since $\zeta \in L^2(\mathbb{R}_+, \mathbb{R}^n)$, it is a meagre function, as defined in Logemann and Ryan [32]. Now this lemma follows from Theorem 4.4 in [32] by taking there $x = \zeta$, $G = \mathbb{R}^n$ and $g(w) = \|w\|$. $\square$

Corollary 6.1.2 *Suppose that $\zeta \in L^2(\mathbb{R}_+,\mathbb{R}^n)$ and $\dot{\zeta} \in L^2(\mathbb{R}_+,\mathbb{R}^n) + L^\infty(\mathbb{R}_+,\mathbb{R}^n)$. Then $\lim\limits_{t\to\infty} \zeta(t) = 0$.*

Indeed, any function ζ as in the corollary is uniformly continuous.

Proposition 6.1.3 *For every $d \in L^2(\mathbb{R}_+,\mathbb{R}^n)$ and for every initial conditions $(e(0),\zeta(0),\tilde{\theta}(0)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$, the state trajectory $(e,\zeta,\tilde{\theta})$ of the closed-loop system in Figure 6.3 is uniquely defined for all $t \geq 0$, bounded and $\lim\limits_{t\to\infty} e(t) = 0$.*

If $r, \dot{r}, \ddot{r}$ are bounded, then also $\lim\limits_{t\to\infty} \zeta(t) = 0$ and $\lim\limits_{t\to\infty} \dot{e}(t) = 0$.

PROOF. Using

$$H_{cl}(e,\zeta,\tilde{\theta},t) = \tilde{H}(e,\zeta,\tilde{\theta},t) + \frac{\mu k}{2}\|e\|^2$$

as a storage function, where $\tilde{H}$ is as in (6.18) and μ is as in (6.7), it can be shown that the closed-loop system is time-varying strictly output passive with input d and output ζ. Indeed, using (6.12), (6.19) and since $v = d - D\zeta$, we have

$$\begin{aligned}
\dot{H}_{cl} &\leq \langle \zeta, d\rangle - k\|\zeta\|^2 + \mu k\langle e, -\Lambda e - \zeta\rangle \\
&\leq \langle \zeta, d\rangle - k\|\zeta\|^2 - \mu^2 k\|e\|^2 - \mu k\langle e,\zeta\rangle \\
&= \langle \zeta, d\rangle - \left\langle \begin{bmatrix}\zeta \\ e\end{bmatrix}, \begin{bmatrix} kI & \mu k/2I \\ \mu k/2I & \mu^2 kI\end{bmatrix}\begin{bmatrix}\zeta \\ e\end{bmatrix}\right\rangle \\
&\leq \langle \zeta, d\rangle - c_1\|\zeta\|^2,
\end{aligned} \tag{6.20}$$

where $c_1 > 0$ satisfies $\begin{bmatrix} kI & \mu k/2I \\ \mu k/2I & \mu^2 kI\end{bmatrix} \geq c_1 I$.

It can be shown that the closed-loop system equations satisfy the Assumptions **(A3)–(A4)** in Section 3.4. Indeed, the closed-loop system equation can be written as

$$\dot{z} = F(t,z,d) = f(t,z) + \begin{bmatrix} 0 \\ (\mathcal{M}_r(e,t))^{-1}(d - D\zeta) \\ 0 \end{bmatrix} \tag{6.21}$$

where $z = \begin{bmatrix} e \\ \zeta \\ \tilde{\theta}\end{bmatrix}$ and f is given by

$$f(t,z) = \begin{bmatrix} -\Lambda e - \zeta \\ (\mathcal{M}_r)^{-1}\left(-\mathcal{D}_r\zeta + \Phi_r\tilde{\theta}\right) \\ -\lambda\Phi_r^T\zeta \end{bmatrix}. \tag{6.22}$$

Using Lemma A.0.2 in Appendix A and since $(\mathcal{M}_r(e,t))^{-1}$ is locally Lipschitz w.r.t. e where the Lipschitz constant depends on $r(t)$, it follows that for any compact set $\mathcal{B} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ there exist locally bounded functions $\alpha, \gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|F(t,z_1,a) - F(t,z_2,a)\| \leq (\alpha(t) + \gamma(t)\|a\|)\,\|z_1 - z_2\|,$$

holds for all $z_1, z_2 \in \mathcal{B}$, $a \in \mathbb{R}^n$ and for all $t \in \mathbb{R}_+$.

It is clear that F is continuous and $(\mathcal{M}_r)^{-1}$ is bounded from above, hence for every $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$, there exists a locally bounded function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $c_2 > 0$ such that

$$\|F(t,b,a)\| \leq \beta(t) + c_2\|a\| \qquad \forall t \in \mathbb{R}_+.$$

These estimates imply that F satisfies Assumptions **(A3)–(A4)** in Section 3.4. Since H_{cl} is proper, it follows from Proposition 3.4.1 that for every $d \in L^2(\mathbb{R}_+, \mathbb{R}^n)$ and for every initial state $(e(0), \zeta(0), \tilde{\theta}(0)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$, there exists a unique global solution of (6.21) and the state trajectory $(e, \zeta, \tilde{\theta})$ is bounded.

Using (6.20) we have that for every $d \in L^2(\mathbb{R}_+, \mathbb{R}^n)$, the signal ζ is in $L^2(\mathbb{R}_+, \mathbb{R}^n)$, see [29, Lemma 6.5]. Since the system (6.12) with input ζ and output e is stable and $\zeta \in L^2(\mathbb{R}_+, \mathbb{R}^n)$, it follows that $e \in L^2(\mathbb{R}_+, \mathbb{R}^n)$. Since $\dot{e} = -\Lambda e - \zeta$, we have $\dot{e} \in L^2(\mathbb{R}_+, \mathbb{R}^n)$. By Corollary 6.1.2 we get that $\lim\limits_{t\to\infty} e(t) = 0$.

Using (6.20) we have that for every $d \in L^2(\mathbb{R}_+, \mathbb{R}^n)$, the signal ζ is in $L^2(\mathbb{R}_+, \mathbb{R}^n)$, see [29, Lemma 6.5]. Since the system (6.12) with input ζ and output e is stable and $\zeta \in L^2(\mathbb{R}_+, \mathbb{R}^n)$, it follows that $e \in L^2(\mathbb{R}_+, \mathbb{R}^n)$. Since $\dot{e} = -\Lambda e - \zeta$, we have $\dot{e} \in L^2(\mathbb{R}_+, \mathbb{R}^n)$. By Corollary 6.1.2 we get that $\lim\limits_{t\to\infty} e(t) = 0$.

We now prove the second part of the proposition, so we assume that $r, \dot{r}, \ddot{r}$ are bounded. We rewrite (6.13) as follows:

$$\dot{\zeta} = \mathcal{M}_r(e,t)^{-1}\left(-\mathcal{D}_r(e,\zeta,t)\zeta + \Phi_r(e,\zeta,t)\tilde{\theta} + d - D\zeta\right). \tag{6.23}$$

It follows from (6.2) that $\mathcal{M}_r(e,t)^{-1} \leq m_1^{-1} I$. Since $e, \zeta, \tilde{\theta}$ and $r, \dot{r}, \ddot{r}$ are bounded, the continuity of $\mathcal{D}$ and Φ implies that $\mathcal{D}_r(e(t), \zeta(t), t)$ and $\Phi_r(e(t), \zeta(t), t)$ are bounded functions of t. Since $\zeta, d \in L^2(\mathbb{R}_+, \mathbb{R}^n)$, it follows that the function

$$G(t) = \mathcal{M}_r(e,t)^{-1}(-D_r(e,\zeta,t)\zeta(t) + d(t) - D\zeta(t))$$

is in $L^2(\mathbb{R}_+, \mathbb{R}^n)$. Since $\tilde{\theta} \in L^\infty(\mathbb{R}_+, \mathbb{R}^n)$, the function $H(t) = \mathcal{M}_r(e,t)^{-1}\Phi_r(e,\zeta,t)\tilde{\theta}(t)$ is in $L^\infty(\mathbb{R}_+, \mathbb{R}^n)$. Thus, (6.23) shows that $\dot{\zeta} \in L^2(\mathbb{R}_+, \mathbb{R}^n) + L^\infty(\mathbb{R}_+, \mathbb{R}^n)$. By Corollary 6.1.2 we have $\lim\limits_{t\to\infty} \zeta(t) = 0$. Now from (6.12) it follows that $\lim\limits_{t\to\infty} \dot{e}(t) = 0$. □

Suppose now that the disturbance d can be decomposed as $d = d_0 + d_E$, where $d_0 \in L^2(\mathbb{R}_+, \mathbb{R}^n)$ and d_E is generated by the exosystem **E** as in (4.1) which is described by

$$\begin{aligned} \dot{w} &= Sw, \\ d_E(t) &= C_w w(t), \end{aligned} \tag{6.24}$$

where $C_w \in \mathbb{R}^{n \times p}$, $w(t) \in \mathbb{R}^p$ is the exosystem state, $S \in \mathbb{R}^{p \times p}$ has its eigenvalues on the imaginary axis and e^{St} is uniformly bounded for $t \geq 0$.

Before the next proposition, it will be convenient to summarize our assumptions. The plant **P** is described by (6.1), with assumptions on the functions $\mathcal{M}, \mathcal{D}$ and g as described there. We have a reference signal $r \in \mathcal{C}^2(\mathbb{R}_+, \mathbb{R}^n)$. The new plant $\tilde{\mathbf{P}}$ from Figure 6.1 is obtained by connecting to **P** a *Slotine-Li* type controller, described by (6.5), (6.6), (6.7) and (6.8). We shall now propose another controller **C**, to be connected to $\tilde{\mathbf{P}}$ as in Figure 6.2(b), in order to reject the disturbance $d = d_0 + d_E$ described above. This controller is based on the internal model principle.

Proposition 6.1.4 *Suppose that $d = d_0 + d_E$, where $d_0 \in L^2(\mathbb{R}_+, \mathbb{R}^n)$ and d_E is generated by the exosystem* **E** *from* (4.1)*. Let the controller* **C** *with state $x_c(t) \in \mathbb{R}^l$, $l \geq p$, be given by*

$$\dot{x}_c = Ax_c - B\zeta, \qquad y_c = B^T x_c - D\zeta, \tag{6.25}$$

where $A^T + A = 0$, (B^T, A) is observable, $D = D^T \geq kI$, $k > 0$ and there exists $\Sigma \in \mathbb{R}^{l\times p}$ such that

$$\Sigma S = A\Sigma, \qquad B^T\Sigma + C_w = 0. \tag{6.26}$$

Then for every initial conditions $(e(0), \zeta(0), \tilde{\theta}(0), x_c(0), w(0)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^p$ and for any $d_0 \in L^2(\mathbb{R}_+, \mathbb{R}^n)$, the state trajectory $(e, \zeta, \tilde{\theta}, x_c)$ of the closed-loop system **L** *from Figure* 6.2(b) *is uniquely defined for all $t \geq 0$, bounded and $\lim_{t\to\infty} e(t) = 0$.*

If $r, \dot{r}, \ddot{r}$ are bounded, then also $\lim_{t\to\infty} \zeta(t) = 0$ and $\lim_{t\to\infty} \dot{e}(t) = 0$.

Note that Proposition 6.1.3 is a particular case of this proposition, corresponding to $p = 0$ (i.e., there is no exosystem).

PROOF. Let us denote $\rho = x_c - \Sigma w$, then from (6.13), (4.1), (6.25), (6.26), (6.14) and (6.12) we have

$$\begin{aligned}
\dot{e} &= -\Lambda e - \zeta, && (6.27)\\
\mathcal{M}_r\dot{\zeta} &= -\mathcal{D}_r\zeta + \Phi_r\tilde{\theta} + B^T(\rho + \Sigma w) + C_w w - D\zeta + d_0 &&\\
&= -\mathcal{D}_r\zeta + \Phi_r\tilde{\theta} + B^T\rho - D\zeta + d_0, && (6.28)\\
\dot{\tilde{\theta}} &= -\lambda\Phi_r^T\zeta, && (6.29)\\
\dot{\rho} &= A(\rho + \Sigma w) - B\zeta - \Sigma S w &&\\
&= A\rho - B\zeta, && (6.30)
\end{aligned}$$

where $(e, \zeta, \tilde{\theta}, \rho)$ is the state of the closed-loop system **L**.

Using the notation $\mathcal{M}_r$ from (6.15), we introduce the storage function

$$H_{cl} = \frac{1}{2}\langle\mathcal{M}_r(e,t)\zeta, \zeta\rangle + \frac{\mu k}{2}\|e\|^2 + \frac{1}{2}\langle\tilde{\theta}, \lambda^{-1}\tilde{\theta}\rangle + \frac{1}{2}\|\rho\|^2$$

for the system **L**. (Note that $H_{cl} = \tilde{H} + \frac{\mu k}{2}\|e\|^2 + \frac{1}{2}\|\rho\|^2$, where $\tilde{H}$ is defined in (6.18) and μ is as in (6.7).) Then using (6.27), (6.28), (6.29) and (6.17),

$$\begin{aligned} \dot{H}_{cl} &= \langle \zeta, -\mathcal{D}_r\zeta + \Phi_r\tilde{\theta} + B^T\rho - D\zeta + d_0\rangle + \frac{1}{2}\langle \dot{\mathcal{M}}_r\zeta, \zeta\rangle \\ &\quad +\mu k\langle e, -\Lambda e - \zeta\rangle + \langle\tilde{\theta}, -\Phi_r^T\zeta\rangle + \langle\rho, A\rho - B\zeta\rangle \\ &\leq \langle\zeta, d_0\rangle - \left\langle \begin{bmatrix}\zeta\\ e\end{bmatrix}, \begin{bmatrix} kI & \mu k/2I \\ \mu k/2I & \mu^2 kI\end{bmatrix}\begin{bmatrix}\zeta \\ e\end{bmatrix}\right\rangle \\ &\leq \langle\zeta, d_0\rangle - c_1\|\zeta\|^2, \end{aligned} \tag{6.31}$$

where $c_1 > 0$ satisfies $\begin{bmatrix} kI & \mu k/2I \\ \mu k/2I & \mu^2 kI\end{bmatrix} \geq c_1 I$.

It can be shown that the closed-loop system equations satisfy the Assumptions **(A3)–(A4)** in Section 3.4. Indeed, the closed-loop system equation can be written as

$$\dot{\tilde{z}} = \tilde{F}(t,\tilde{z},d_0) = \begin{bmatrix} f(t,z) \\ 0\end{bmatrix} + \begin{bmatrix} 0 \\ (\mathcal{M}_r(e,t))^{-1}(d_0 - D\zeta) \\ 0 \\ A\rho - B\zeta\end{bmatrix},$$

where $\tilde{z} = \begin{bmatrix} e \\ \zeta \\ \tilde{\theta} \\ \rho\end{bmatrix}$ and f is as in (6.22).

Using Lemma A.0.2 in Appendix A and since $(\mathcal{M}_r(e,t))^{-1}$ is locally Lipschitz, it can be checked that for any compact set $\mathcal{B} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l$ there exist locally bounded functions $\alpha, \gamma : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|\tilde{F}(t,z_1,a) - \tilde{F}(t,z_2,a)\| \leq (\alpha(t) + \gamma(t)\|a\|)\,\|z_1 - z_2\|$$

holds for all $z_1, z_2 \in \mathcal{B}$, $a \in \mathbb{R}^n$ and for all $t \in \mathbb{R}_+$.

It is clear that $\tilde{F}$ is continuous and $(\mathcal{M}_r)^{-1}$ is bounded from above, hence for every $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$, there exists a locally bounded function $\beta : \mathbb{R}_+ \to \mathbb{R}_+$ and a constant $c_2 > 0$ such that

$$\|\tilde{F}(t,b,a)\| \leq \beta(t) + c_2\|a\| \qquad \forall t \in \mathbb{R}_+.$$

These imply that the function $\tilde{F}$ satisfies Assumptions **(A3)–(A4)** in Section 3.4. Since H_{cl} is proper, it follows from Proposition 3.4.1 that for any $d_0 \in L^2(\mathbb{R}_+, \mathbb{R}^n)$ and for any initial state $(e(0), \zeta(0), \tilde{\theta}(0), x_c(0), w(0))$, there exists a unique global solution $(e, \zeta, \tilde{\theta}, \rho)$ of the closed-loop system and the state trajectory $(e, \zeta, \tilde{\theta}, \rho)$ is bounded. Since ρ and w are bounded and $\rho = x_c - \Sigma w$, we see that x_c is also bounded.

Using (6.31), $d_0 \in L^2(\mathbb{R}_+, \mathbb{R}^n)$ implies that also $\zeta \in L^2(\mathbb{R}_+, \mathbb{R}^n)$. We can now argue exactly as in the proof of Proposition 6.1.3 to conclude that $\lim_{t\to\infty} e(t) = 0$.

To prove the last part of the proposition, we assume that $r, \dot{r}, \ddot{r}$ are bounded. We rewrite (6.27) as follows:

$$\dot{\zeta} = \mathcal{M}_r^{-1}\left(-\mathcal{D}_r\zeta + \Phi_r\tilde{\theta} + B^T\rho - D\zeta + d_0\right). \tag{6.32}$$

It follows from (6.2) that $\mathcal{M}_r(e,t)^{-1} \leq m_1^{-1}I$. Since $e, \zeta, \tilde{\theta}$ and $r, \dot{r}, \ddot{r}$ are bounded, the continuity of $\mathcal{D}$ and Φ implies that $\mathcal{D}_r(e(t),\zeta(t),t)$ and $\Phi_r(e(t),\zeta(t),t)$ are bounded functions of t. Since $\zeta, d_0 \in L^2(\mathbb{R}_+,\mathbb{R}^n)$, it follows that the function

$$G(t) = \mathcal{M}_r(e,t)^{-1}[-\mathcal{D}_r(e,\zeta,t)\zeta(t) + d_0(t) - D\zeta(t)]$$

is in $L^2(\mathbb{R}_+,\mathbb{R}^n)$. Since $\tilde{\theta} \in L^\infty(\mathbb{R}_+,\mathbb{R}^n)$ and $\rho \in L^\infty(\mathbb{R}_+,\mathbb{R}^l)$, the function

$$H(t) = \mathcal{M}_r(e,t)^{-1}\left[\Phi_r(e,\zeta,t)\tilde{\theta}(t) + B^T\rho(t)\right]$$

is in $L^\infty(\mathbb{R}_+,\mathbb{R}^n)$. Thus, (6.32) shows that $\dot{\zeta} \in L^2(\mathbb{R}_+,\mathbb{R}^n) + L^\infty(\mathbb{R}_+,\mathbb{R}^n)$. By Corollary 6.1.2 we obtain $\lim_{t\to\infty}\zeta(t) = 0$. Now from (6.12) it follows that $\lim_{t\to\infty}\dot{e}(t) = 0$. □

The following theorem, which is our main result, is just a reformulation of Proposition 6.1.4 for bounded $r, \dot{r}, \ddot{r}$, in terms of the original plant $\mathbf{P}$.

Theorem 6.1.5 *Consider the system* $\mathbf{P}$ *as in* (6.1) *with outputs q and $\dot{q}$, the reference $r \in \mathcal{C}^2(\mathbb{R}_+,\mathbb{R}^n)$, where $r, \dot{r}, \ddot{r}$ are bounded and the disturbance $d = d_0 + d_E$, where $d_0 \in L^2(\mathbb{R}_+,\mathbb{R}^n)$ and d_E is generated by the exosystem* $\mathbf{E}$ *from* (4.1). *Let the controller* $\tilde{\mathbf{C}}$ *be given by the state equations*

$$\begin{aligned} \dot{x}_c &= Ax_c - B\zeta, & (6.33)\\ \dot{\hat{\theta}} &= -\lambda\Phi(q,\dot{q},\dot{\xi},\xi)^T\zeta, & (6.34) \end{aligned}$$

where $x_c(t) \in \mathbb{R}^l$, $l \geq p$, $e = r - q$, $\zeta(t) \in \mathbb{R}^n$ is defined in (6.12), *$A \in \mathbb{R}^{l\times l}$, $A^T + A = 0$, $B \in \mathbb{R}^{l\times n}$. Here, $\hat{\theta}(t) \in \mathbb{R}^m$, $\xi = \dot{r} + \Lambda e$, $\lambda \in \mathbb{R}^{m\times m}$, $\lambda^T = \lambda > 0$ and Φ is as in* (6.5). *The controller generates the signal*

$$z = \hat{\mathcal{M}}\dot{\xi} + \hat{\mathcal{D}}\xi + \hat{g} + B^T x_c - D\zeta,$$

where $z(t) \in \mathbb{R}^n$ and $D = D^T \geq kI$, $k > 0$. We assume that (B^T, A) is observable and there exists $\Sigma \in \mathbb{R}^{l\times p}$ which satisfies (6.26).

Then for every $(q(0),\dot{q}(0),\hat{\theta}(0),x_c(0),w(0)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^l \times \mathbb{R}^p$ and for every $d_0 \in L^2(\mathbb{R}_+,\mathbb{R}^n)$, the state trajectory $(q,\dot{q},\hat{\theta},x_c)$ of the closed-loop system $\mathbf{L}$ *shown in Figure* 6.2(a) *is uniquely defined for all $t \geq 0$, bounded, $\lim_{t\to\infty} e(t) = 0$ and $\lim_{t\to\infty}\dot{e}(t) = 0$.*

PROOF. The closed-loop system **L** from Figure 6.2(a) is the same as the one in Figure 6.2(b). The state variables for Figure 6.2(a) can be expressed in terms of those for Figure 6.2(b) as follows:

$$\begin{aligned} q &= r - e, \\ \dot{q} &= \dot{r} + \zeta + \Lambda e, \\ \hat{\theta} &= \theta + \tilde{\theta}. \end{aligned} \tag{6.35}$$

Now all the claims follow from Proposition 6.1.4. □

6.2 High-order controller design

We have seen in Theorem 6.1.5 that the internal model based compensator described by (6.33) solves the tracking and disturbance rejection problem for **P** if the equations (6.26) have a solution. However, this result by itself does not indicate any practical way to construct A and B, since C_w and S are not known (we only know the frequencies of the signal). Motivated by the result in Chapter 4, we use the result from Theorem 4.4.3 to construct A and B using only the eigenvalues of S as our input data (these correspond to the frequencies of the exosystems **E**).

Theorem 6.2.1 *Consider the system* **P** *as in* (6.1) *with outputs q and $\dot{q}$ and the reference $r \in \mathcal{C}^2(\mathbb{R}_+, \mathbb{R}^n)$, where $r, \dot{r}, \ddot{r}$ are bounded. Suppose that the disturbance $d = d_0 + d_E$, where $d_0 \in L^2(\mathbb{R}_+, \mathbb{R}^n)$, d_E is generated by the exosystem* **E** *from* (4.1) *and denote by χ the minimal polynomial of S. Let the controller $\tilde{\mathbf{C}}$ be given by the state equations*

$$\dot{x}_c = Ax_c - B\zeta, \tag{6.36}$$

$$\dot{\hat{\theta}} = -\lambda \Phi(q, \dot{q}, \dot{\xi}, \xi)^T \zeta, \tag{6.37}$$

where $x_c(t) \in \mathbb{R}^{\kappa n}$ (κ is the degree of χ), $e = r - q$, $\hat{\theta}(t) \in \mathbb{R}^m$, $\xi = \dot{r} + \Lambda e$, $\zeta(t) \in \mathbb{R}^n$ is defined in (6.12), *$\lambda \in \mathbb{R}^{m\times m}$, $\lambda^T = \lambda > 0$ and Φ is as in* (6.5). *The matrices $A \in \mathbb{R}^{\kappa n\times \kappa n}$ and $B \in \mathbb{R}^{\kappa n \times n}$ are*

$$A = \begin{bmatrix} S_{min} & 0 & \cdots & 0 \\ 0 & S_{min} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & S_{min} \end{bmatrix}, \qquad B = \begin{bmatrix} \Gamma_1 & 0 & \cdots & 0 \\ 0 & \Gamma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Gamma_n \end{bmatrix}, \tag{6.38}$$

where S_{min} has characteristic polynomial χ and satisfies $S_{min} + S_{min}^T = 0$, and (Γ_i^T, S_{min}) is observable for $i = 1, \ldots n$. Let the controller output be given by

$$z = \hat{\mathcal{M}}\dot{\xi} + \hat{\mathcal{D}}\xi + \hat{g} + B^T x_c - D\zeta,$$

where $z(t) \in \mathbb{R}^n$ *and* $D = D^T \geq kI$, $k > 0$.

Then for every $(q(0), \dot{q}(0), \hat{\theta}(0), x_c(0), w(0)) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{\kappa n} \times \mathbb{R}^p$ *and for every* $d_0 \in L^2(\mathbb{R}_+, \mathbb{R}^n)$*, the state trajectory* $(q, \dot{q}, \hat{\theta}, x_c)$ *of the closed-loop system* $\mathbf{L}$ *shown in Figure 6.2*(a) *is uniquely defined for all* $t \geq 0$*, bounded,* $\lim_{t\to\infty} e(t) = 0$ *and* $\lim_{t\to\infty} \dot{e}(t) = 0$.

PROOF. The theorem is proved by showing that there exists a mapping Σ satisfying (6.26) and then using Theorem 6.1.5. Let ϕ_c^r and ϕ_w^r be given by

$$\phi_c^r = \begin{bmatrix} B^T \\ B^T A \\ \vdots \\ B^T A^{\kappa-1} \end{bmatrix} \qquad \phi_w^r = \begin{bmatrix} C_w \\ C_w S \\ \vdots \\ C_w S^{\kappa-1} \end{bmatrix}. \tag{6.39}$$

It can checked that $\phi_c^r \in \mathbb{R}^{\kappa m \times \kappa m}$ from (6.39) has full rank, so that it is invertible (as shown in Chapter 4). Let

$$\Sigma = -(\phi_c^r)^{-1} \phi_w^r. \tag{6.40}$$

By simple algebraic manipulations (as shown in Theorem 4.4.3), it can be shown that Σ in (6.40) satisfies (6.26). $\square$

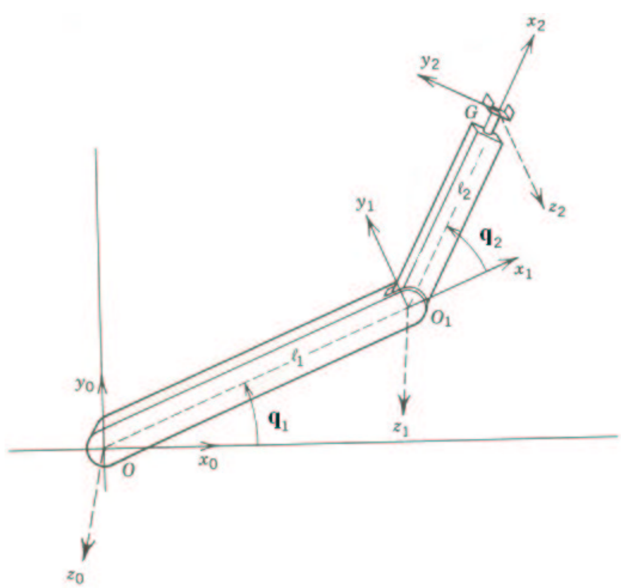


Figure 6.4: The planar manipulator with two degrees of freedom used in our simulation. This picture is taken from [30].

6.3 A simulation result

A two-link planar manipulator shown in Figure 6.4 is used in the simulation. The system is modeled as in (6.1), with the link angles as the generalized coordinates, $q = \begin{bmatrix} q_1 & q_2 \end{bmatrix}^T$. The inertia matrix $\mathcal{M}(q)$, as derived in [30], is given by

$$\mathcal{M}(q) = \begin{bmatrix} \mathcal{M}_{11} & \mathcal{M}_{12} \\ \mathcal{M}_{21} & \mathcal{M}_{22} \end{bmatrix},$$

with

$$\begin{aligned} \mathcal{M}_{11} &= I_{o1} + m_1 l_1^2/4 + I_{o2} + m_2 l_2^2/4 + m_2 l_1^2 + m_2 l_1 l_2 \cos q_2, \\ \mathcal{M}_{12} &= \mathcal{M}_{21} = I_{o2} + m_2 l_2^2/4 + m_2 l_1 l_2 \cos q_2/2, \\ \mathcal{M}_{22} &= I_{o2} + m_2 l_2^2/4, \end{aligned}$$

where I_{oi} denotes the inertia of link i in kg m^2, m_i denotes the mass of link i in kg, l_i denotes the length of link i in m and q_i denotes the angle of link i in rad. The Coriolis torque matrix $\mathcal{D}(q, \dot{q})$ is given by

$$\mathcal{D}(q, \dot{q}) = \begin{bmatrix} \mathcal{D}_{11} & \mathcal{D}_{12} \\ \mathcal{D}_{21} & \mathcal{D}_{22} \end{bmatrix},$$

with

$$\begin{aligned} \mathcal{D}_{11} &= -(m_2 l_1 l_2 \sin q_2/2)\dot{q}_2, \\ \mathcal{D}_{12} &= -(m_2 l_1 l_2 \sin q_2/2)(\dot{q}_1 + \dot{q}_2), \\ \mathcal{D}_{21} &= (m_2 l_1 l_2 \sin q_2/2)\dot{q}_1, \\ \mathcal{D}_{22} &= 0. \end{aligned}$$

The gravity vector $g(q)$ is given by

$$g(q) = \begin{bmatrix} m_1 g_0 l_1 \cos q_1/2 + m_2 g_0 l_1 \cos q_1 + m_2 g_0 l_2 \cos(q_1 + q_2)/2 \\ m_2 g_0 l_2 \cos(q_1 + q_2) \end{bmatrix},$$

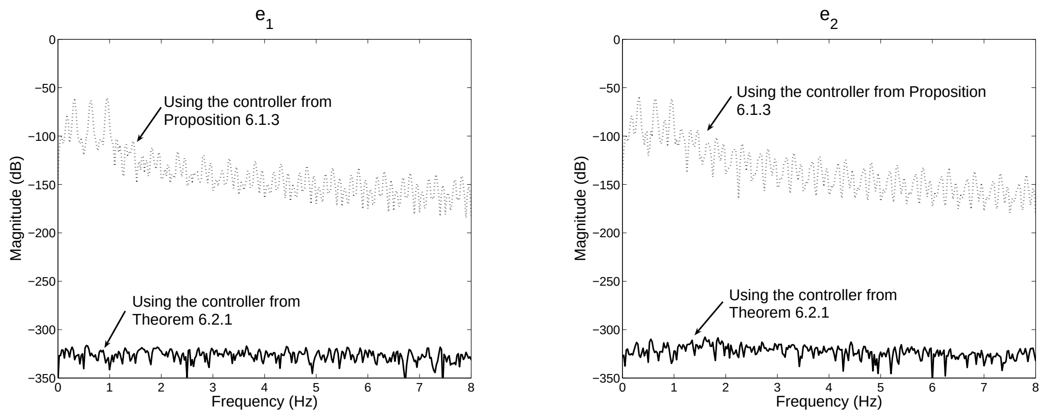
where g_0 is the gravitational acceleration in m/s^2.

Using the two-link planar manipulator model above, we consider the following vector θ of unknown parameters:

$$\begin{aligned} \theta_1 &= I_{o1} + m_1 l_1^2/4 + I_{o2} + m_2 l_2^2/4 + m_2 l_1^2, \\ \theta_2 &= m_2 l_1 l_2, \\ \theta_3 &= I_{o2} + m_2 l_2^2/4, \\ \theta_4 &= m_1 g_0 l_1/2 + m_2 g_0 l_1, \\ \theta_5 &= m_2 g_0 l_2/2, \end{aligned} \tag{6.41}$$

and the corresponding $\Phi(q, \dot{q}, \dot{\xi}, \xi)$ can be obtained from (6.5).

We assume that $m_1 = 9.8$kg, $m_2 = 10.2$kg, $l_1 = 0.49$m, $l_2 = 0.51$m, $g_0 = 9.85$m/s^2, $I_{o1} = 1.4125$kg m^2 and $I_{o2} = 1.4208$kg m^2. Here we have assumed that 1kg motors are installed at both ends of each spherical rods. For the initial estimate $\hat{\theta}(0)$, we

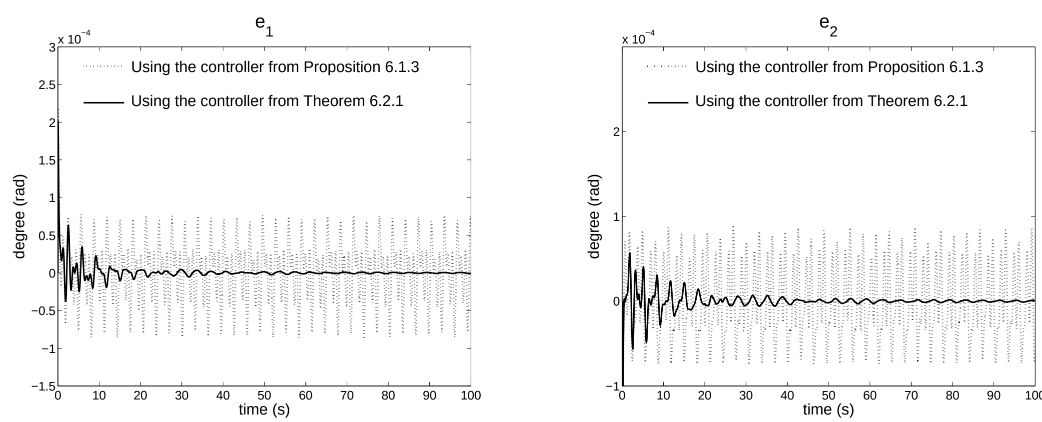


(a) (b)

Figure 6.5: (a). The power spectral density of the tracking error e_1 using the controller from Proposition 6.1.3 with $y_c = -D\zeta$ (upper curve) and using the controller from Theorem 6.2.1 (lower curve) computed for the time interval $[1960, 2000]$s. (b). The same as (a) but for e_2.

use the following numerical values: $\hat{m}_1 = 10$kg, $\hat{m}_2 = 10$kg, $\hat{l}_1 = 0.5$m, $\hat{l}_2 = 0.5$m, $\hat{g}_0 = 9.8$m/s^2, $\hat{I}_{o1} = 1.4$kg m^2 and $\hat{I}_{o2} = 1.4$kg m^2.

The reference signal for q_1, denoted by r_1, is produced by twice integrating a triangular wave, with a period of 4s. This produces a $\mathcal{C}^2$ periodic signal with a fundamental angular frequency of $\omega = \pi/2$. The reference signal for q_2 is $r_2 = -r_1$. Thus, if the manipulator tracks the reference, then it mimics a "waiter's hand" movement, meaning that the second link remains horizontal all the time.



(a) (b)

Figure 6.6: (a). The tracking error e_1 using the controller from Proposition 6.1.3 with $y_c = -D\zeta$ (dotted line) and using the controller from Theorem 6.2.1 (solid line). (b). The same as (a) but for e_2.

The input disturbance signal $d(t) = d_E(t) \in \mathbb{R}^2$ is generated by (4.1), where

$$S = \begin{bmatrix} 0 & -\Omega \\ \Omega & 0 \end{bmatrix}, \quad \Omega = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix},$$

$$C_w = \begin{bmatrix} 0.5 & 0.25 & 0.125 & 0 & 0 & 0 \\ 1 & 0.5 & -0.5 & 0.25 & -0.25 & 0.125 \end{bmatrix},$$

$$w(0) = \begin{bmatrix} 0.7537 & -0.9742 & -0.3792 & 0.5582 & -0.3854 & 0.8534 \end{bmatrix}^T.$$

The parameters for the controller as in Theorem 6.2.1 are given by $\Lambda = 100$, $\lambda = 100I$, $S_{min} = S$, $\Gamma_i = 10\begin{bmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}^T$ for $i = 1,2$ and $D = 100I$. For comparison, we have also simulated the tracking controller from Proposition 6.1.3, with the same D, Λ and λ.

After a long simulation time we have computed the power spectral density of e using the windowed PSD algorithm from Matlab (we have used the window from 1960s to 2000s). It can be seen from Figure 6.5 that when the controller uses only the feedback (6.6) with proportional gain $y_c = -D\zeta$ as in Proposition 6.1.3, then the tracking errors are small but not negligible. From the plots in Figure 6.5, we also see that the controller discussed in Theorem 6.2.1 is able to reject the disturbance d. Indeed, the power spectral densities of e_1 and e_2 using the controller from Theorem 6.2.1 are practically zero. In Figure 6.6 we see that the controller **C** from Theorem 6.2.1 makes the error signals converge to zero, as expected.

For the simulation using the controller from Theorem 6.2.1, Figure 6.7 shows the evolution of the parameter estimates (the components of $\hat{\theta}$). It is interesting to see that the signals $\hat{\theta}_1$ and $\hat{\theta}_3$ converge to different values from the actual θ_1 and θ_3 but the tracking error signal e still converges to zero.

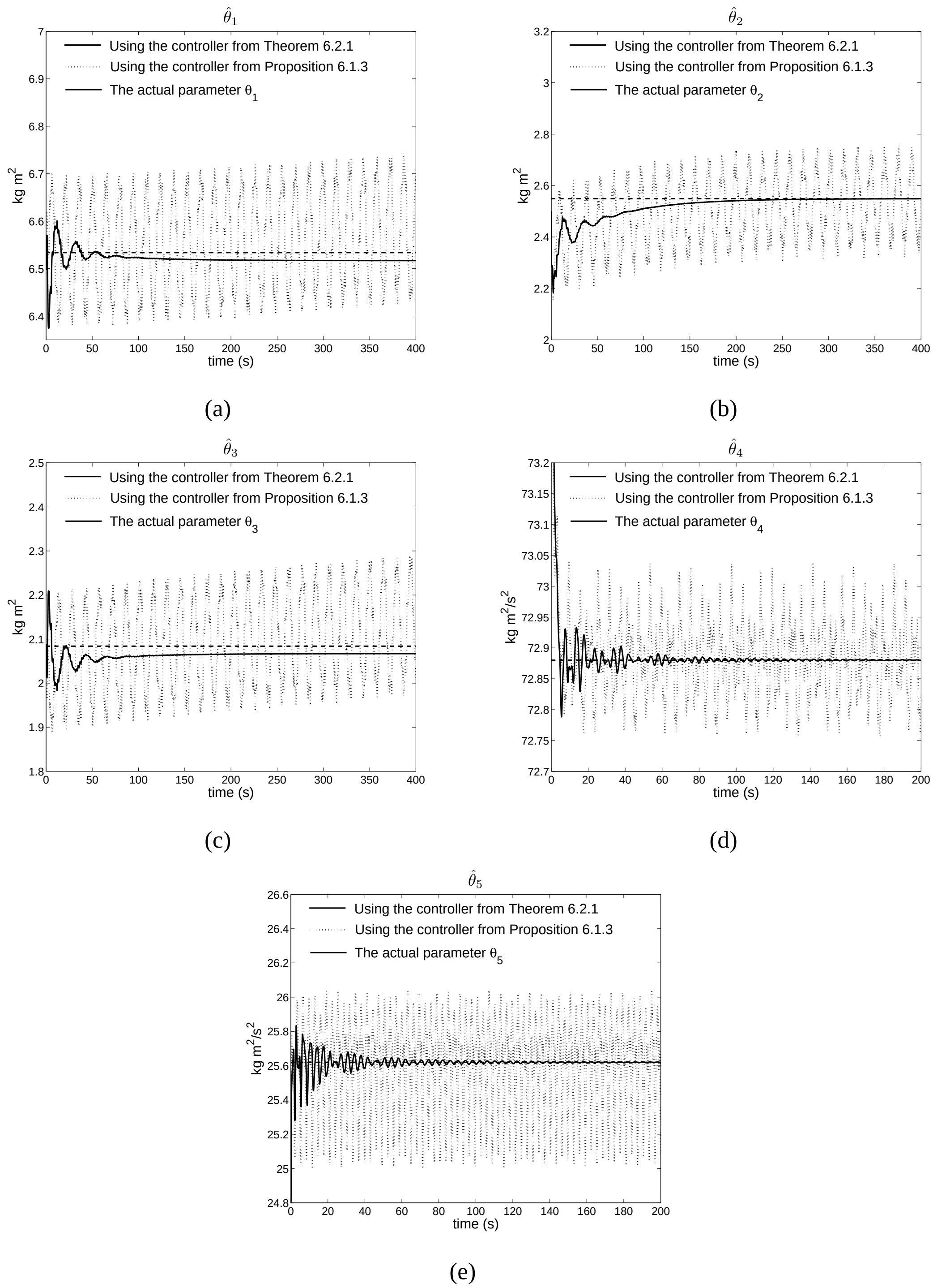


Figure 6.7: The plot of $\hat{\theta}$ using the controller from Proposition 6.1.3 with $y_c = -D\zeta$ (dotted line), using the controller from Theorem 6.2.1 (solid line) and the true value of θ (dashed line). (a) $\hat{\theta}_1$; (b) $\hat{\theta}_2$; (c) $\hat{\theta}_3$; (d) $\hat{\theta}_4$; (e) $\hat{\theta}_5$.

Chapter 7

Conclusions and Future Works

7.1 Conclusions

It has been shown that under mild assumptions on the differential equation, if the system is zero-state detectable, strictly output passive and its storage function H is proper, then the state x converges to zero. This implies the existence of global solution of the system for any input $u \in L^2_{loc}$. A notion of L^2 system-stability is introduced which has several interesting interconnection properties in the study of nonlinear systems.

A simple LTI controller has been proposed to solve the input disturbance rejection problem for passive nonlinear plants, where the disturbance can be decomposed into a finite superposition of sine waves of arbitrary but known frequencies and an L^2 signal. A necessary and sufficient condition is given for the LTI controller to solve the input disturbance rejection problem.

The tracking of constant reference signal in passive system is studied for a class of passive system called constant incremental passive systems. This class of systems allows us to solve the tracking problem for constant signals by recasting it into an input disturbance rejection problem for constant signal.

For a fully actuated mechanical system, the tracking of a $\mathcal{C}^2$ reference signal and rejecting a disturbance signal d can be solved by combining a Slotine-Li type adaptive controller with an LTI controller.

7.2 Future works

Several possible extensions of the results presented in this thesis have been identified. They are listed as follows:

1. Passive systems with hysteresis is a possible generalization of the works in this thesis. A motivation to generalize the passive nonlinear systems discussed in the thesis to include a hysteresis operator comes from an article by Gorbet *et al* in [14]. The paper [14] shows that systems with Preisach-type hysteresis operator have passivity property. Therefore it is expected that the results given in this thesis may include passive nonlinear systems which contains also hysteresis operator. Our preliminary study shows that the conjecture holds for systems with Preisach hysteresis operator. A thorough analysis in the modeling of hysteresis operator can be found in the book by Mayergoyz [36].

2. It has been shown in this thesis that tracking of non-constant reference signal can be achieved for incremental passive systems. Yet nonlinear systems which belongs to the incremental passive systems has not been thoroughly investigated.

3. The technique of proving the state convergence using infinite-dimensional systems theory may become a useful tools to show the boundedness and the convergence of state in various problems in nonlinear systems theory and applications that involves L^p input signals where $p \in [1,\infty)$. It serves as an alternative tool to the Barbălat's lemma in nonlinear system analysis. For example, suppose that we have the problem of state estimation of the plant $\mathbf{P}$ described by

$$\begin{aligned} \dot{x} &= f(x,u), \\ y &= h(x), \end{aligned}$$

where $x(t) \in \mathbb{R}^n, u \in L^p(\mathbb{R}_+,\mathbb{R}^m), y(t) \in \mathbb{R}^p$ are the state, the input and the measurement output. The state estimator $\mathbf{Q}$ is given by

$$\dot{\hat{x}} = g(\hat{x},y,u),$$

where $\hat{x}(t) \in \mathbb{R}^n$. By using a coordinate transformation $\tilde{x} = \hat{x} - x$, the state equations of $\mathbf{P}$ and $\mathbf{Q}$ can be written as follows

$$\begin{aligned} \dot{x} &= f(x,u), \\ \dot{\tilde{x}} &= g(\tilde{x}+x,h(x),u) - f(x,u). \end{aligned}$$

Hence the design of the state estimator $\mathbf{Q}$ is to find a suitable functions g,γ and Lyapunov function $H(x,\tilde{x})$ such that $\dot{H} \leq \|u\|^2 - k\|\gamma(\tilde{x})\|^2$, $k > 0$ and the following detectability condition of $\mathbf{P}$ and $\mathbf{Q}$ is satisfied: $\gamma(\tilde{x}(t)) = 0$ and $u(t) = 0$ for all $t \geq 0 \ \Rightarrow \tilde{x}(t) \to 0$ as $t \to \infty$.

Appendix A

Condition (A3) for fully-actuated mechanical systems with *Slotine-Li* controller

We shall show that the condition **(A3)** in Section 3.4 is satisfied for the time-varying system $\tilde{\mathbf{P}}$ obtained from the fully actuated mechanical system (6.1) with the *Slotine-Li* controller.

Lemma A.0.1 *Suppose that* $f_1 \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^{p_1 \times p_2})$ *and* $f_2 \in \mathcal{C}(\mathbb{R}^n, \mathbb{R}^{p_2 \times p_3})$ *are locally Lipschitz. Then the function* $f : x \mapsto f_1(x) f_2(x)$ *is locally Lipschitz.*

PROOF. By assumptions, for any compact set $\mathcal{B} \in \mathbb{R}^n$ there exists constant $L > 0$ such that

$$\begin{aligned} \|f_1(a) - f_1(b)\| &\leq L\|a-b\|, \\ \|f_2(a) - f_2(b)\| &\leq L\|a-b\| \end{aligned}$$

hold for all $a, b \in \mathcal{B}$. Then

$$\begin{aligned} \|f(a) - f(b)\| &= \|f_1(a) f_2(a) - f_1(a) f_2(b) + f_1(a) f_2(b) - f_1(b) f_2(b)\| \\ &\leq \|f_1(a)(f_2(a) - f_2(b))\| + \|(f_1(a) - f_1(b)) f_2(b)\| \\ &\leq L \left(\max_{a \in \mathcal{B}} \|f_1(a)\| + \max_{b \in \mathcal{B}} \|f_2(b)\| \right) \|a-b\|, \end{aligned}$$

holds for all $a, b \in \mathcal{B}$. $\square$

Let us recall the state equations of $\tilde{\mathbf{P}}$ from Chapter 6 and the assumptions stated there:

$$\dot{z} = f(t,z) + \begin{bmatrix} 0 \\ (\mathcal{M}_r(e,t))^{-1} v \\ 0 \end{bmatrix}, \tag{A.1}$$

where $z(t) = \left[\begin{smallmatrix} e(t) \\ \zeta(t) \\ \tilde{\theta}(t) \end{smallmatrix}\right] \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$ and f is given by

$$f(t,z) = \begin{bmatrix} -\Lambda e - \zeta \\ (\mathcal{M}_r)^{-1}(-\mathcal{D}_r \zeta + \Phi_r \tilde{\theta}) \\ -\lambda \Phi_r^T \zeta \end{bmatrix}.$$

Remember that $\mathcal{M}_r, \mathcal{D}_r$ and Φ_r are defined in terms of $\mathcal{M}, \mathcal{D}$ and g, see (6.15), where $\mathcal{M} \in \mathcal{C}^1$ and $\mathcal{D}, g$ are locally Lipschitz. The reference signal r is assumed to be in $\mathcal{C}^2$. It follows that $\mathcal{M}_r \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}^n)$ and $\mathcal{D}_r, \Phi_r \in \mathcal{C}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}^n)$. We have $m_1 I \leq \mathcal{M}_r(e,t) \leq m_2 I$ for all $(e,t) \in \mathbb{R}^n \times \mathbb{R}_+$ where $m_1, m_2 > 0$, according to (6.2). It follows that

$$\frac{1}{m_2} I \leq (\mathcal{M}_r(e,t))^{-1} \leq \frac{1}{m_1} I$$

for all $(e,t) \in \mathbb{R}^n \times \mathbb{R}$ and $(\mathcal{M}_r)^{-1} \in \mathcal{C}^1(\mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}^n)$.

Lemma A.0.2 *For any compact set $\mathcal{B} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$, there exists a locally bounded function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$\|f(t,z_1) - f(t,z_2)\| \leq \alpha(t) \|z_1 - z_2\|$$

holds for all $z_1, z_2 \in \mathcal{B}$ and for all $t \in \mathbb{R}_+$.

PROOF. Since $r \in \mathcal{C}^2(\mathbb{R}_+, \mathbb{R}^n)$, the signals $r, \dot{r}$ and $\ddot{r}$ are locally bounded and continuous.

Take any compact set $\mathcal{B} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m$. Using the local Lipschitz assumption on $\mathcal{D}$ and using Lemma A.0.1, it is easy to see that there exists a locally bounded function $\gamma_1 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\begin{aligned} &\|\mathcal{D}_r(e_1, \zeta_1, t)\zeta_1 - \mathcal{D}_r(e_2, \zeta_2, t)\zeta_2\| \\ &= \|\mathcal{D}(r(t) - e_1, \dot{r}(t) + \zeta_1 + \Lambda e_1)\zeta_1 \\ &\quad - \mathcal{D}(r(t) - e_2, \dot{r}(t) + \zeta_2 + \Lambda e_2)\zeta_2\| \\ &\leq \gamma_1(t) \left\| \left[\begin{smallmatrix} e_1 - e_2 \\ \zeta_1 - \zeta_2 \end{smallmatrix}\right] \right\| \qquad t \in \mathbb{R}_+ \end{aligned} \tag{A.2}$$

holds for all $z_1 = \begin{bmatrix} e_1 \\ \zeta_1 \\ \tilde{\theta}_1 \end{bmatrix}, z_2 = \begin{bmatrix} e_2 \\ \zeta_2 \\ \tilde{\theta}_2 \end{bmatrix} \in \mathcal{B}$.

Denote $\Phi_i(q,\dot{q},a,b) = \mathcal{M}_i(q)a + \mathcal{D}_i(q,\dot{q})b + g_i(q)$, $i = 1,2,\ldots m$ where $\mathcal{M}_i, \mathcal{D}_i$ and g_i are as in (6.4). Using the local Lipschitz assumption on $\mathcal{M}_i, \mathcal{D}_i$ and g_i and using Lemma A.0.1 we conclude that Φ_i is locally Lipschitz.

As in (6.5), denote $\Phi = \begin{bmatrix} \Phi_1 & \Phi_2 & \cdots & \Phi_m \end{bmatrix}$. Since Φ_i is locally Lipschitz for all $i = 1,2\ldots m$ and $r, \dot{r}, \ddot{r}$ are locally bounded, it can be checked that there exists a locally bounded function $\gamma_2 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|\Phi_r(e_1,\zeta_1,t)\tilde{\theta}_1 - \Phi_r(e_2,\zeta_2,t)\tilde{\theta}_2\| \leq \gamma_2(t)\,\|z_1 - z_2\| \tag{A.3}$$

holds for all $z_1, z_2 \in \mathcal{B}$ and for all $t \in \mathbb{R}_+$. This condition is also satisfied for $\Phi_r(e,\zeta,t)^T\zeta$ with a locally bounded function $\gamma_3 : \mathbb{R}_+ \to \mathbb{R}_+$.

Denote

$$\Psi(z,t) = \mathcal{M}_r^{-1}(e,t)\left(-\mathcal{D}_r(e,\zeta,t)\zeta + \Phi_r(e,\zeta,t)\tilde{\theta}\right).$$

Note that $(\mathcal{M}_r(e,t))^{-1}$ is locally Lipschitz w.r.t. e, with the Lipschitz constant depending on $r(t)$. Using this fact, (A.2), (A.3) and Lemma A.0.1, it follows that there exists a locally bounded function $\gamma_4 : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\|\Psi(e_1,\zeta_1,\tilde{\theta}_1,t) - \Psi(e_2,\zeta_2,\tilde{\theta}_2,t)\| \quad \leq \quad \gamma_4(t)\,\|z_1 - z_2\|$$

holds for all $z_1, z_2 \in \mathcal{B}$ and for all $t \in \mathbb{R}_+$.

It is easy to see that there exists a positive constant $c_2 > 0$ such that

$$\|(-\Lambda e_1 - \zeta_1) - (-\Lambda e_2 - \zeta_2)\| \leq c_2 \left\| \begin{bmatrix} e_1 - e_2 \\ \zeta_1 - \zeta_2 \end{bmatrix} \right\|$$

holds for all $z_1, z_2 \in \mathcal{B}$.

The lemma is proved with $\alpha(t) = \lambda\gamma_3(t) + \gamma_4(t) + c_2$. $\square$

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