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An invitation to algebraic number theory and class field theory

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Abstract

This informal document was motivated by a question here at my university by a bachelor student. I will try to expose something that personally I think is impressive. The aim is to do it in such a way that is understandable with a basic knowledge of algebra.

We will examine without any rigor the "generalization" in some sense of the concept of "factorization", not just in the integers $\mathbb{Z}$ but in some "generalized" integers from number fields of the form $\mathbb{Q}(\sqrt{-n})$ for $n > 0$ which are called imaginary quadratic number fields. The problems in these new rings that will arise will give us interesting mathematics.

Contents

0.1 Rings ......................................................... 3
0.2 Ideals of a Ring ............................................. 3
0.3 Proving that an ideal of a ring of integers is not principal .... 4
0.4 Operations with ideals of a ring .............................. 5
0.5 Extending the ideals to fractional ideals ...................... 6
0.6 Class group of a ring of integers ........................... 7
0.7 Weird and impressive facts about $\text{Cl}(R)$ and conclusion ... 7
First of all we will assume that the rings that we will use here are com-
mutative and they will have a multiplicative neutral element 1.
We begin with basics, most of the people here are familiar with the ring of
integers:
\[ \mathbb{Z} = \{ \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots \} \]
We know that in \( \mathbb{Z} \) are prime numbers and composite numbers.

Then we have the field \( \mathbb{Q} \) with all the "fractions" in it gotten from numbers
of \( \mathbb{Z} \). In \( \mathbb{Q} \) numbers like \( \pi, e \) are missing. There are just numbers that can
be represented as \( \frac{a}{b} \) such that \( a, b \in \mathbb{Z} \).

The most usual ring of integers is of course \( \mathbb{Z} \), but we can add a new alge-
braic element in it and experiment with this new space. For example if we
add the root of a polynomial which is not rational (an algebraic number);
and then we calculate all the possible combinations of it with the field \( \mathbb{Q} \),
we will get a new field.

For example if we adjoin one of the roots of \( x^2 + 5 \) which is \( \sqrt{-5} = i\sqrt{5} \)
where \( i \) is the imaginary unit we get the imaginary quadratic field:
\[ \mathbb{Q}(\sqrt{-5}) = \{ a + b\sqrt{-5} : a, b \in \mathbb{Q} \} \]
(1)

This space is like the usual \( \mathbb{Q} \), but we added a new element not in \( \mathbb{Q} \) (the
algebraic number \( \sqrt{-5} \)). Is easy to see that every nonzero element there has
inverse and that the product of two elements also belongs to \( \mathbb{Q}(\sqrt{-5}) \).
Take \( a + b\sqrt{-5}, c + d\sqrt{-5} \in \mathbb{Q}(\sqrt{-5}) \) then the product of both numbers is
given by \( ac - 5bd + (ad + bc)\sqrt{-5} \in \mathbb{Q}(\sqrt{-5}) \) and the inverse of \( a + b\sqrt{-5} \)
is given by:
\[ \frac{1}{a + b\sqrt{-5}} = \frac{a}{a^2 + 5b^2} + \frac{-b\sqrt{-5}}{a^2 + 5b^2} \in \mathbb{Q}(\sqrt{-5}) \]

So, \( \mathbb{Q}(\sqrt{-5}) \) is a well defined field, just as \( \mathbb{Q} \).

The ring of integers which will be the equivalent to \( \mathbb{Z} \) in \( \mathbb{Q} \), but now for
\( \mathbb{Q}(\sqrt{-5}) \) is:
\[ \mathbb{Z}[\sqrt{-5}] = \{ a + b\sqrt{-5} : a, b \in \mathbb{Z} \} \]
(2)
Is well known that the usual ring of integers \( \mathbb{Z} \) is a Unique Factorization
Domain, this means that any number can be factored in prime numbers of
\( \mathbb{Z} \) in a unique way.
For a general number field (extensions of $\mathbb{Q}$ by other algebraic elements like our example) this is not always true, in our example $\mathbb{Z}[\sqrt{-5}]$ is not a unique factorization domain.

Example:

$$6 = 3 \cdot 2 = (1 + \sqrt{-5})(1 - \sqrt{-5})$$  \hspace{1cm} (3)

So we will see what is happening here, but first we will need some theory, I begin with the essential, and finish with the deep theory.

0.1 Rings

We know that a general ring has two operations $\langle R, +, \cdot \rangle$ where with the addition $\langle R, + \rangle$ all elements have an additive inverse, a neutral element and here particularly is commutative. Also it has a product $\langle R, \cdot \rangle$ which interacts via the law of distributivity with the addition. Here in $R$ not necessary every element is invertible.

The common example is again $\langle \mathbb{Z}, +, \cdot \rangle$ because the multiplicative inverse of the nonzero $a \in \mathbb{Z}$ is an integer $a^{-1} \in \mathbb{Z}$ such that $a \cdot a^{-1} = 1$, such number does not exist in $\mathbb{Z}$ unless $a = \pm 1$ because there is no $\frac{1}{a} \in \mathbb{Z}$ for $a \neq \pm 1$. For the addition for each integer $a$ there is always a ”negative” $-a$ such that under $+$ it gives us 0, this means $a + (-a) = 0$ (neutral element).

0.2 Ideals of a Ring

A ring $\langle R, +, \cdot \rangle$ is an algebraic structure that has some subalgebraic structures called ideals. If $I \subset R$ is an ideal then their elements form a group under $+$, namely $\langle I, + \rangle \leq \langle R, + \rangle$ and for $a \in I$ multiplication by every element $r \in R$, namely $a \cdot i \in I$, that is $aI = I$.

For example in $\mathbb{Z}$ consider the ”multiples of $n$” there are the ideals and we denote them as $n\mathbb{Z} := \{n \cdot a : a \in \mathbb{Z}\}$.

The reason that $n\mathbb{Z}$ is closed under addition is rather obvious as $a, b \in n\mathbb{Z}$ implies that $a + b = n\alpha + n\beta = n(\alpha + \beta) \in n\mathbb{Z}$ for some $\alpha, \beta \in \mathbb{Z}$. The other property of the ideal is also obvious as for $r \in \mathbb{Z}$ and $a \in n\mathbb{Z}$ we have that $ra = nar \in n\mathbb{Z}$ for some $\alpha \in \mathbb{Z}$.

Said this, these ideals $I \subset R$ behave more-less like a vector space over $R$ (an $R$-module), as $rI = I$ for all $r \in R$, just like in $\mathbb{R}^2$ which can be seen as a $\mathbb{R}$-vector space, as any element $v \in \mathbb{R}^2$, can be multiplied by an $\alpha \in \mathbb{R}$.
such that $\alpha v \in \mathbb{R}^2$. Note that if $1 \in I$ then $I = R$, so much more interesting examples arise when the ideal is not the ring, which is our case.

So now, we said that the ideals $I \subset R$ behave "like a vector space", well then we can think in a "basis" of them, which means, a set of elements that generate the ideal. This is denoted by $I = \langle a, b \rangle$ If you just need 1 element we call the ideal principal, that is $I = \langle a \rangle = aR$ for some $a \in R$.

In $\mathbb{Z}$ all ideals are principal.

So, now $\mathbb{Z}$ is starting to be boring, because an ideal of $\mathbb{Z}$ given by $\langle m, n \rangle$ consists in all the linear combinations $xm + yn$ and is generated by one element.

More precisely $\langle m, n \rangle = \langle \gcd(m, n) \rangle = \gcd(m, n)\mathbb{Z}$.

The proof that every ideal in $\mathbb{Z}$ is principal can be seen here for the curious [https://proofwiki.org/wiki/Ring_of_Integers_is_Principal_Ideal_Domain](https://proofwiki.org/wiki/Ring_of_Integers_is_Principal_Ideal_Domain).

Returning to our example with the ring of integers $\mathbb{Z}[\sqrt{-5}]$ of the field $\mathbb{Q}(\sqrt{-5})$, there are ideals, like $I = \langle 3, 1 + \sqrt{-5} \rangle$ that are not principal.

Principal ideals play a crucial role in unique factorization of $R$ as we will see in the next theorem.

**Theorem 1:**

Let $K = \mathbb{Q}(\alpha)$ be an algebraic number field and $R$ its ring of integers, then:

All ideals of $R$ are principal if and only if $R$ has unique factorization for its elements.

We will prove in the next section that $I = \langle 3, 1 + \sqrt{-5} \rangle$ is not principal and hence, using the latter theorem $\mathbb{Z}[\sqrt{-5}]$ is not a unique factorization domain. We already knew that it was not a factorization domain as 6 as an element of $\mathbb{Z}[\sqrt{-5}]$ has two different non trivial factorizations.

The next section can be skipped as it has more technicalities, but is just to fill a big hole that a curious person may find in this informal text.

### 0.3 Proving that an ideal of a ring of integers is not principal

Recall that this is informal, so I will give the essential ideas.

The basic tool to factorize in the ring of integers $\mathbb{Z}[\sqrt{D}]$ (where $D$ is non
square) is the following:

**Definition:** The norm of a quadratic field of the form $\mathbb{Q}(\sqrt{D})$ for a non-square $D$ is given by the function $N : \mathbb{Q}(\sqrt{D}) \to \mathbb{Q}$ such that:

$$N(a + b\sqrt{D}) = (a + b\sqrt{D})(a - b\sqrt{D}) = a^2 - Db^2$$

The following property is the key of why the norm is important to factorize when applied to elements $\mathbb{Z}[\sqrt{D}]$.

**Proposition 2:** Let $\alpha, \beta \in \mathbb{Q}(\sqrt{D})$ then if $N$ is its associated norm then $N(\alpha \beta) = N(\alpha) N(\beta)$.

Now using this proposition the next theorem tells us information about the "dimension" of the ideal of a number field in general, i.e. the number of generators of the ideal.

**Theorem 3:** Let $R$ be a ring of integers of a number field $K$, consider $z \in R$ and the principal ideal generated by $z$, namely $(z) = zR$ then $N(z) = \#R/(z)$

With this theorem 3 now we can prove something about the factorization of the integers in $K = \mathbb{Q}(\sqrt{-5})$ when we combine it with theorem 2. So here $R = \mathbb{Z}[\sqrt{-5}]$ and $N(a + b\sqrt{-5}) = a^2 + 5b^2$.

We want to prove that $I = \langle 3, 1 + \sqrt{-5} \rangle$ is not principal. So using the previous theorem this reduces to prove that $\#R/I \neq N(z)$ for all $z \in R$. This is easy as $R/(3) \cong \mathbb{Z}/3\mathbb{Z}[\sqrt{-5}]$ then $\#R/(3) = 9$ and this implies that for the quotient with the bigger ideal $I$ we have $\#R/I = 3$, then we would like an element $a + b\sqrt{-5} \in \mathbb{Z}[\sqrt{-5}]$ with norm 3, which means we want an integer solution to $a^2 + 5b^2 = 3$ which is impossible. and then $I$ cannot be principal.

### 0.4 Operations with ideals of a ring

Consider now set of all ideals $I \subseteq R$ (counting the trivial ideal $R$), call them $\mathfrak{I}$, now if you take $I, J \in \mathfrak{I}$ there is a way of multiply them $K = I \otimes J$ in such a way that $K \in \mathfrak{I}$, that is, $K$ is also an ideal of $(R, +, \cdot)$.

We define

$$I \otimes J := \left\{ \sum_{i=1}^{n} a_i \cdot b_i : a_i \in I, b_i \in J \forall n \in \mathbb{N} \right\}$$

This new ideal $I \otimes J$ is intuitively all the possible products $ab$ that can be constructed for $a \in I$ and $b \in J$. 


Example: Consider the ring of polynomials in 4 variables with complex coefficients \( R = \mathbb{C}[x, y, z, w] \) and consider the following ideals generated by different combinations of variables \( I = (z, w), \ J = (x + z, y + w), \ K = (x + z, w) \). Then:

\[
IJ = (z(x + z), z(y + w), w(x + z), w(y + w)) = (z^2 + xz, zy + wy, wx + wz, wy + w^2)
\]

\[
IK = (xz + z^2, zw, xw + zw, w^2)
\]

Now we need to drop redundant ideals in \( \mathfrak{I} \) so we define an equivalence relation \( \sim \) in the ideals of \( \mathfrak{I} \), namely \( I \sim J \) if there are \( a, b \in R \) such that \( aI = bJ \). So here, ideals which are related by \( \sim \) are denoted as \([I]\). If you choose a representative of \([I]\) and \([J]\) then \([I] \otimes [J] = [IJ]\) is well defined and commutative.

So here we have a new structure \( \mathfrak{I}/\sim \) which has multiplicative structure with the ideals.

### 0.5 Extending the ideals to fractional ideals

Here comes an interesting part, we are limiting ourselves to rings of the form \( \mathbb{Z}[\sqrt{D}] \) for a non square \( D \) which are the integers of the quadratic field \( \mathbb{Q}(\sqrt{D}) \).

Now we would like to define for an ideal \( I \subset \mathbb{Z}[\sqrt{D}] \) another ideal \( I^{-1} \) which we will call fractional Ideal, that will give us an "identity", the identity will be the boring ideal we mentioned before, which is the whole \( R \). So we are trying to find a good structure for \( I^{-1} \) such that \( I \oplus I^{-1} = R \).

This \( I^{-1} \) will depend now in the elements of \( \mathbb{Q}(\sqrt{D}) \), as we need some "fractions", to "clear" denominators and get the 1 \( I \otimes I^{-1} \) which makes an ideal trivial, i.e. it makes it \( R \).

\[
I^{-1} := \{ z \in \mathbb{Q}(\sqrt{D}) : z \cdot I \subseteq \mathbb{Z}[\sqrt{D}] \}
\]

This is exactly as we need, as the \( z \) are the "fractions" in the fields where multiplied by \( I \) stays in the integer ring \( \mathbb{Z}[\sqrt{D}] \). Then we have that \( I \otimes I^{-1} = R \).

This fractional ideals are not exactly ideals of \( \mathbb{Z}[\sqrt{D}] \) as they are not elements of \( \mathbb{Z}[\sqrt{D}] \), they are \( \mathbb{Z}[\sqrt{D}] \)-submodules of the field \( \mathbb{Q}(\sqrt{D}) \) but this
is a matter of terminology.

A big remark is that for a general integer ring $R$ is not always possible to invert the ideals, but in our world of number fields $\mathbb{Q}(\sqrt{\alpha})$ where $\alpha$ is an algebraic integer, the ideals of the corresponding integer ring are **always** invertible.

Here we are studying Dedekind Rings which means that every ideal $I \subset \mathbb{Z}[\sqrt{D}]$ can be factored by prime ideals using the multiplication $\otimes$. This means that it is always possible to find a **unique finite set** of prime ideals $\{P_i\}$ of $\mathbb{Z}[\sqrt{D}]$ such that $I = \otimes P_i$.

### 0.6 Class group of a ring of integers

The final object is built from the multiplicative structure given before, namely by all the ideals $I \subseteq R$ with the equivalence relation $\sim$, and with multiplication of ideals $\otimes$ namely $(I/\sim, \otimes)$.

Now extend $I/\sim$ to all the fractional its ideals (we can do this as we are supposing that $R$ is a Dedekind domain), call this extension to fractional ideals $\mathfrak{F}$.

Consider now the same equivalence classes of $\mathfrak{F}$ using $\sim$ defined in the previous section.

We define the Class group of $R$ by

$$\text{Cl}(R) := \langle \mathfrak{F}/\sim, \otimes \rangle$$

**Theorem 4:** Let $R$ be a ring of integers of a number field $K$ then $\#\text{Cl}(R) = 1$ if and only if $R$ has unique factorization

### 0.7 Weird and impressive facts about $\text{Cl}(R)$ and conclusion

The class group $\text{Cl}(R)$ as the theorem 4 says, has only 1 element then the ring $R$ has unique factorization in its elements (not only on ideals), so it measures in some way how complicated the factorization behaves in a ring $R$.

A Weird and impressive thing is that for rings of the form $\mathbb{Z}[\sqrt{-n}]$ there is only a finite number of values for $n$ where $\mathbb{Z}[\sqrt{-n}]$ has Unique factorization of its elements (not only in on the ideals). This is the same as saying
that $\#\text{Cl}(\mathbb{Z}[\sqrt{-n}]) = 1$ by theorem 4. The numbers for $n$ such that we get Unique factorization are: \{1, 2, 3, 7, 11, 19, 43, 67, 163\}.

These weird numbers that generate Unique factorization in their associated imaginary quadratic fields are called \textbf{Heegner numbers}, and our example $\mathbb{Z}[\sqrt{-5}]$ as we saw does not have Unique factorization as 5 is not a Heegner number, and in fact $\#\text{Cl}(\mathbb{Z}[\sqrt{-5}]) = 2$.

These Heegner numbers also are related to something that could be a mathematical coincidence but, it isn’t, and is the fact that the numbers $e^{\pi \sqrt{43}}, e^{\pi \sqrt{67}}, e^{\pi \sqrt{163}}$ are ”almost an integer”:

\[
e^{\pi \sqrt{43}} \approx 884736743.999777466
\]
\[
e^{\pi \sqrt{67}} \approx 147197952743.999998662454
\]
\[
e^{\pi \sqrt{163}} \approx 262537412640768743.9999999999925007
\]

This is not a coincidence but proving it, requires a little more analytic theory and the theory of Eisenstein series, but I hope this move your mind to be more interested in algebraic number theory.