Full order observer design for a class of port-Hamiltonian systems with dissipation

"The definition of a good mathematical problem is the mathematics it generates rather than the problem itself." - Andrew Wiles.

In this chapter, we consider a special class of port-Hamiltonian systems with dissipation (PHSD) for which we propose a design methodology for constructing globally exponentially stable full-order observers by using a passivity-based approach (refer to [77]). The essential idea is to make the augmented system consisting of the plant and the observer dynamics to become strictly passive with respect to an invariant manifold defined on the extended state space on which the state estimation error is zero. We first introduce the concept of passivity of a system with respect to a manifold by defining a new input and output on the extended state space and then perform a partial state feedback passivation which leads to the construction of the observer. We then illustrate this observer design procedure for some well known mechanical and electro-mechanical systems, modeled in the form of a PHSD. We also prove under some additional assumptions the separation principle for the proposed observer, when employed in closed-loop with a passivity based control (PBC) state feedback law, by using concepts from nonlinear cascaded systems theory.

The observer design theory which we present in this chapter shares a similar philosophy with the I & I observer proposed in Chapter 3. The underlying idea there was to select a manifold in the extended state space of the plant and observer and make it positively invariant and attractive with respect to the plant-observer dynamics. In this chapter, we aim to make the augmented system passive with respect to the manifold by defining a new input-output pair.
4. Full Order Observer Design for a class of PHSD

4.1 Passivity based observers for port-Hamiltonian systems with dissipation

We consider the following special class of port-Hamiltonian systems with dissipation,

\[
\begin{bmatrix}
\dot{x}_1 \\
\dot{x}_2
\end{bmatrix} = \begin{bmatrix}
J_1(x_1, u_1) - R_1(x_1) & T(x_1, u_1) \\
-T^T(x_1, u_1) & J_2(x_1, u_1) - R_2(x_1)
\end{bmatrix} \begin{bmatrix}
\nabla x_1 H(x) \\
\nabla x_2 H(x)
\end{bmatrix} + \begin{bmatrix}
g_1(x_1) \\
g_2(x_1)
\end{bmatrix} u_2,
\]

(4.1)

where \(x = (x_1, x_2)\) where \(x_1 \in \mathbb{R}^n, x_2 \in \mathbb{R}^p\) are the states, \(u_1 \in \mathcal{U} \subset \mathbb{R}^m\), \(u_2 \in \mathbb{R}^m\) are the inputs where \(\mathcal{U}\) is a compact set. We consider only \(x_1\) to be measurable, that is, the measured output is \(y = x_1\) (which may or may not equal the standard PHSD output \(y_p = g^T(x) \nabla x H(x)\)). The matrices \(J_1 \in \mathbb{R}^{n \times n}, J_2 \in \mathbb{R}^{p \times p}\) are skew-symmetric, \(R_1 \in \mathbb{R}^{n \times n}, R_2 \in \mathbb{R}^{p \times p}\) are symmetric positive semi-definite and further \(T \in \mathbb{R}^{n \times p}, g_1 \in \mathbb{R}^{n \times m}, g_2 \in \mathbb{R}^{p \times m}\). We assume each of the matrices \(J_1, J_2, R_1, R_2, T, g_1, g_2\) to be smooth in their arguments. The Hamiltonian \(H : \mathbb{R}^n \times \mathbb{R}^p \rightarrow \mathbb{R}\) assumes the form

\[
H(x_1, x_2) = x_2^T Q x_2 + K(x_1),
\]

(4.2)

where \(Q^T = Q > 0\) is a constant matrix and \(K\) is a smooth nonlinear function of \(x_1\). This implies that the dynamics is affine in the unmeasured state \(x_2\). Although the above class of systems seems rather restricted, it does encompass a good number of physical examples as illustrated later. It can be seen that the class of partially linearizable systems \(S_{PLVCC}\) considered in Chapter 3 are of the form (4.1) with \(x_1 = q\) and \(x_2 = P\).

We now proceed to design under certain assumptions, a globally exponentially stable full order observer for the system (4.1).

4.1.1 Problem Formulation

We start by defining a passivity based observer for the system (4.1) in which we introduce the concept of strict passivity of a system with respect to a manifold.

**Definition 4.1.** We call the dynamical system represented as

\[
\begin{bmatrix}
\dot{\hat{x}}_1 \\
\dot{\hat{x}}_2
\end{bmatrix} = \begin{bmatrix}
J_1(\hat{x}_1, u_1) - R_1(\hat{x}_1) & T(\hat{x}_1, u_1) \\
-T^T(\hat{x}_1, u_1) & J_2(\hat{x}_1, u_1) - R_2(\hat{x}_1)
\end{bmatrix} \begin{bmatrix}
\nabla \hat{x}_1 H(\hat{x}) \\
\nabla \hat{x}_2 H(\hat{x})
\end{bmatrix} + \begin{bmatrix}
g_1(x_1) \\
g_2(x_1)
\end{bmatrix} u_2 + \begin{bmatrix}
L_1(\hat{x}_1) \\
L_2(\hat{x}_1)
\end{bmatrix} v,
\]

(4.3)
4.1 Observers and alternate passive input-output pairs for PHSD

where \( \dot{x} = (\dot{x}_1, \dot{x}_2) \), \( \dot{x}_1 \in \mathbb{R}^n, \dot{x}_2 \in \mathbb{R}^p, v \in \mathbb{R}^n \), a passivity based observer for the system (4.1) if there exists smooth matrices \( L_1 : \mathbb{R}^n \to \mathbb{R}^{n \times n}, L_2 : \mathbb{R}^n \to \mathbb{R}^{p \times n} \) and a continuous scalar function \( k : \mathbb{R}^p \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) such that the feedback law

\[
v = L_1^{-1}(\dot{x}_1)X^{-1}\{k(y, \dot{x}, u_1)y_d + v_d\}, \tag{4.4}
\]

with \( \bar{X} = \bar{X}^T > 0 \), makes the augmented system\(^1\) composed of (4.1) and (4.3) strictly passive with respect to the manifold

\[
\mathcal{M} = \{(x, \dot{x}) : x = \dot{x}\}, \tag{4.5}
\]

from the new input \( v_d \) to the new output \( y_d = \dot{x}_1 - x_1 \).

**Definition 4.2.** The system (4.1), (4.3) with the design input (4.4) is strictly passive with respect to the manifold \( \mathcal{M} \) (from input \( v_d \) to output \( y_d \)), uniformly for all \( u_1 \in \mathcal{U} \) and \( u_2 \in \mathbb{R}^m \) if there exists a storage function \( S(x, \dot{x}) > 0 \) for every \( x \neq \ddot{x} \), \( S(x, \dot{x}) = 0 \) on \( \mathcal{M} \) and the time derivative of \( S \) along the system trajectory satisfies:

\[
\nabla_x^T S(x, \dot{x})[J(x, u_1) - R(x_1)]\nabla_x H(x) + \nabla_{\dot{x}}^T S(x, \dot{x})[J(\dot{x}_1, u_1) - R(\dot{x}_1)]\nabla_{\dot{x}} (\ddot{x}) + k(y, \dot{x}, u_1)\nabla_{\dot{x}}^T S(x, \dot{x})L(\ddot{x}_1)L_1^{-1}(\dot{x}_1)\bar{X}^{-1}y_d \leq -\alpha_P(\|x - \dot{x}\|), \tag{4.6}
\]

\[
\{\nabla_x^T S(x, \dot{x}) - \nabla_{\dot{x}}^T S(x, \dot{x})\}g(y) \equiv 0, \tag{4.7}
\]

\[
\nabla_{\dot{x}}^T S(x, \dot{x})L(\ddot{x}_1)L_1^{-1}(\dot{x}_1)\bar{X}^{-1} = y_d^T, \tag{4.8}
\]

where \( \alpha_P \) is a positive definite function.

If the augmented system is strictly passive with respect to \( \mathcal{M} \) for some functions \( L_1, L_2 \) and \( k \), then upon letting \( v_d = 0 \) the manifold \( \mathcal{M} \) becomes positively invariant and globally attractive. The solution of the observer design problem then follows by noting that the state estimation error is zero on \( \mathcal{M} \). Further, the asymptotic estimate of \( x \) is then given by \( \dot{x} \).

### 4.1.2 Observer Design

The notion of passivity\(^2\) is usually associated with respect to a point in the state space rather than a manifold as above. For this standard notion of passivity with respect to a point, necessary and sufficient conditions have been

---

\(^1\)In the sequel we shall always use the term *augmented system* to refer to the system composed of (4.1) and (4.3).

\(^2\)A dynamical system \( \dot{z} = f_1(z, u), y = f_2(z, u) \) with state vector \( z \), input \( u \) and output \( y \) is said to be passive with respect to \( u \) and \( y \) if there exists a positive definite storage function \( S \) (i.e., \( S > 0 \) whenever \( z \neq 0 \) and \( S(0) = 0 \)) that satisfies \( S \leq u^Ty \).
established for feedback equivalence of a nonlinear system to a passive system. In [21] it has been shown that any affine control system can be rendered strictly passive (with respect to a point) by a smooth static state feedback if and only if the system has a vector relative degree \( \{1, \ldots, 1\} \) and is globally minimum phase. In situations where some of the states are not measurable, additional sufficiency conditions have been proposed in [41] which ensure feedback passivation by a static output feedback while reference [77] gives sufficient conditions for rendering a system strictly passive with respect to a set, by a partial state feedback. Our situation is similar to [41] and [77] as we need to achieve strict passivity of the augmented system with respect to \( M \) by using a feedback law \( v \) which is independent of \( x_2 \).

We now state two key assumptions on (4.1), (4.3) and use them to prove that:

1. There exist matrices \( L_1(\hat{x}_1) \) and \( L_2(\hat{x}_1) \) such that the augmented system satisfies vector relative degree and global minimum phase conditions with respect to the manifold \( M \) which are analogous to the conditions needed for static state feedback passivation.

2. The augmented system satisfies an additional nonlinear growth inequality which is sufficient to make it strictly passive with respect to \( M \) by a partial state feedback law \( v = L_1^{-1}(\hat{x}_1) \bar{X}^{-1}\{k(y, \hat{x}, u_1)y_d + v_d\} \), which is independent of \( x_2 \).

We start by stating the following assumptions.

**Assumption 3.** There exists a smooth globally invertible matrix \( L_1(x_1) \in \mathbb{R}^{n \times n} \) and a smooth matrix \( L_2(x_1) \in \mathbb{R}^{p \times n} \) such that

\[
A^T(x_1, u_1) + A(x_1, u_1) > \epsilon I_{n \times n}, \quad \epsilon > 0
\]

holds for all \( x_1 \), uniformly for all \( u_1 \in U \), where

\[
A(x_1, u_1) := \{L_2(x_1)L_1^{-1}(x_1)T(x_1, u_1) + R_2(x_1)\}.
\]

**Assumption 4.** There exists a smooth function \( \beta : \mathbb{R}^n \rightarrow \mathbb{R}^p \) such that

\[
L_2(x_1)L_1^{-1}(x_1) = \nabla \beta(x_1)
\]

holds for all \( x_1 \in \mathbb{R}^n \).

Note that for \( p = 1 \), Assumption 4 is always satisfied. We next state the following theorem.

**Theorem 4.3.** Under assumption 3,

1. The augmented system (4.1)-(4.3) has a vector relative degree \( \{1, \ldots, 1\} \) with respect to the input \( v \) and the output \( y_d = \hat{x}_1 - x_1 \).
2. The zero dynamics of the augmented system with respect to the output $y_d$ renders the manifold
\[ P = \{(x_1, x_2, \hat{x}_2) : \hat{x}_2 = x_2\} \] (4.11)
positively invariant and globally exponentially attractive.

**Proof.** We compute the derivative of $y_d$ and see that the input $v$ appears in it pre-multiplied by the matrix $L_1$. From Assumption 3, since $L_1$ is invertible for all $x_1$, we conclude that the augmented system has a vector relative degree \{1, ..., 1\} with respect to the input $v$ and the output $y_d$.

We next see that the zero dynamics of the augmented system with respect to the output $y_d$, defined uniformly for all $u_1 \in U$, $u_2 \in \mathbb{R}^m$, essentially consists of (4.1) and the equations
\[
\begin{aligned}
0 &= T(x_1, u_1)Q\{\hat{x}_2 - x_2\} + L_1(x_1)v, \quad (4.12) \\
\dot{x}_2 &= \{J_2(x_1, u_1) - R_2(x_1)\}Q\hat{x}_2 - T^T(x_1, u_1)\nabla K(x_1) \\
&\quad + g_2(y)u_2 + L_2(x_1)v, \quad (4.13)
\end{aligned}
\]
where we make use of (4.2). We now consider the manifold $\mathcal{P}$ defined in (4.11) and denote its off-the-manifold coordinate as $z = \hat{x}_2 - x_2$. Computing the derivative of $z$ along (4.1), (4.13) and using (4.12) yields
\[ \dot{z} = \{J_2(x_1, u_1) - A(x_1, u_1)\}Qz. \] (4.14)

We can clearly see from (4.14) that the manifold $\mathcal{P}$ is positively invariant and further if we consider the Lyapunov function $V = \frac{1}{2}z^TQz$, then Assumption 3 verifies $\dot{V} \leq -\frac{\lambda_m^2(Q)}{\lambda_M(Q)}V$ with $\lambda_m(Q)$, $\lambda_M(Q)$ denoting the minimum and maximum eigenvalue of the matrix $Q$. Thus $V$ exponentially decays to zero with convergence rate $\frac{\lambda_m^2(Q)}{\lambda_M(Q)}$. \hfill \Box

An interesting corollary that follows from Theorem 4.3 is

**Corollary 4.4.** Under Assumption 3 and the additional Assumption 4, the dynamical system
\[
\begin{aligned}
\dot{\eta} &= -\nabla \beta(x_1)\{[J_1(x_1, u_1) - R_1(x_1)]\nabla K(x_1) + g_1(y)u_2\} \\
&\quad + \{J_2(x_1, u_1) - R_2(x_1) - \nabla \beta(x_1)T(x_1, u_1)\}Q\{\eta + \beta(x_1)\} \\
&\quad - T^T(x_1, u_1)\nabla K(x_1) + g_2(y)u_2 \quad (4.15) \\
\dot{x}_2 &= \eta + \beta(x_1), \quad (4.16)
\end{aligned}
\]
where $\eta \in \mathbb{R}^p$, is a reduced order observer for $x_2$ and the dynamics of $x_1, x_2, \eta$, renders the manifold $\mathcal{N} = \{(x_1, x_2, \eta) : \eta = x_2 - \beta(x_1)\}$ positively invariant and globally exponentially attractive. The asymptotic estimate of $x_2$ is $\eta + \beta(x_1)$. \hfill \Box

77
4. Full Order Observer Design for a class of PHSD

**Proof.** The off-the-manifold coordinate is \( z = \eta - x_2 + \beta(x_1) \), which upon differentiating along the system dynamics yields

\[
\dot{z} = \{J_2(x_1, u_1) - R_2(x_1) - \nabla \beta(x_1)T(x_1, u_1)\}Qz. \tag{4.17}
\]

Using Assumptions 3, 4 and employing the Lyapunov function \( V = \frac{1}{2}z^TQz \), we can prove that (4.17) is globally exponentially stable. The asymptotic estimate of \( x_2 \) is then \( \eta + \beta(x_1) \).

**Remark 4.5.** The reduced-order observer in (4.15) is similar to the \( I & I \) observer which was studied in Chapter 3. Moreover, for S\(_{PLvCC}\) systems the reduced-order observer in (4.15) and the \( I & I \) observer are the same.

**Remark 4.6.** The notion of vector relative degree is usually defined with respect to the output and the total input of the system, which for our augmented system (4.1)-(4.3) would be \( u_1, u_2 \) and \( v \). However, our idea is to design the input \( v \) by the feedback law (4.4) such that the augmented system becomes strictly passive with respect to the the input \( v_d \) and the output \( y_d \), uniformly for all \( u_1 \in U \subset \mathbb{R}^m \) and \( u_2 \in \mathbb{R}^m \). In other words, we consider \( v \) as our design input and the other inputs \( u_1, u_2 \) can be any functions of time or state or both, belonging to their respective domains. Hence, we use the concept of vector relative degree between \( y_d \) and \( v \), which is a small modification of the definition that is usually found in the literature [76].

**Remark 4.7.** The zero dynamics of the augmented system with respect to the output \( y_d \) given by (4.1), (4.12), (4.13), differs slightly from the usual understanding of zero dynamics in the sense that the inputs \( u_1, u_2 \) still remain in our equations. Once again, as already stated in the previous remark, we consider \( v \) to be the design input and define the zero dynamics uniformly for all \( u_1 \in U \subset \mathbb{R}^m \) and \( u_2 \in \mathbb{R}^m \).

**Remark 4.8.** Assumption 3 involves finding matrices \( L_1(x_1), L_2(x_1) \) such that the augmented system has a vector relative degree \( \{1, \ldots, 1\} \) and is globally minimum phase with respect to \( M \) while Assumption 4 states that the quantity \( L_2(x_1)L_1^{-1}(x_1) \) has to be integrable and satisfy (4.10) for some function \( \beta(x_1) \). Designing such state dependent matrices that satisfy (4.9) and (4.10) would involve solving a set of algebraic and partial differential equations respectively and is usually a difficult task. Reference [77] studies the observer design problem by restricting \( L_1, L_2 \) to be constant matrices in which case Assumption 4 is trivially satisfied with \( \beta(x_1) = L_2L_1^{-1}x_1 \) and hence narrows the applicable class of nonlinear systems. Indeed, as we show later in our examples, whenever \( T \) is a constant matrix, letting \( L_1, L_2 \) to be constant would suffice for the observer design, whereas in situations where \( T \) depends on \( x_1 \), it is natural to allow \( L_1, L_2 \) to be state dependent in order to satisfy Assumption 3.
Remark 4.9. Another interesting special situation is when the damping matrix \( R(x_1) > 0 \), in which case the Assumption 3 gets satisfied with \( L_2 = 0 \) and hence the resulting reduced order observer for \( x_2 \) exactly emulates the \( x_2 \) dynamics. However, the convergence rate of \( z \) in (4.17) would then solely depend on the natural damping of the system which could sometimes be very negligible or could be subject to high uncertainties in which case such a reduced-order observer is not generally preferred.

At this stage, we assume that the matrices \( L_1(x_1), L_2(x_1) \) can be designed to satisfy Assumptions 3 and 4. We next state a theorem to prove that the augmented system admits a partial state feedback \( v \), which is independent of \( x_2 \), that renders the system strictly passive with respect to the manifold \( \mathcal{M} \) and also leads to the construction of the full-order observer.

**Theorem 4.10.** Under Assumptions 3 and 4,
1. The system (4.1), (4.3) expressed in the coordinates \((x_1, x_2, \zeta_1, \zeta_2)\) defined as
   \[
   \begin{align*}
   \zeta_1 &= \dot{x}_1 - x_1, \\
   \zeta_2 &= \dot{x}_2 - x_2 - \{\beta(\dot{x}_1) - \beta(x_1)\}
   \end{align*}
   \] (4.18) (4.19)
assumes the global normal form.\(^3\)
2. Under the additional assumption that \( g_1 \equiv 0 \) in (4.1), there exists non-negative scalar functions \( f_1(\zeta_1, \dot{\zeta}_1, x_2, u_1), f_2(\zeta_1, \dot{\zeta}_1, x_2, u_1) \) such that the feedback law
   \[
   v = L_1^{-1}(\dot{x}_1)\bar{X}^{-1}\{-[\delta + f_1 + f_2^2]\zeta_1 + v_d\},
   \] (4.22)
where \( \bar{X} \in \mathbb{R}^{n \times n} = \bar{X}^T > 0, \delta > 0, \) makes the system strictly passive with respect to the manifold \( \mathcal{M} \), uniformly for all \( u_1 \in U, u_2 \in \mathbb{R}^m, \) from the input \( v_d \) to the output \( \zeta_1 \) with the storage function being given by \( W(\zeta) = \frac{1}{2}\zeta_1^TQ\zeta_2 + \frac{1}{2}\zeta_1^T\bar{X}\zeta_1. \)

**Proof.** We first begin by defining the functions \( F_i(\zeta_1, \zeta_2, x_1, x_2, u_1), i = 1, 2, 3, \) with \( F_1 \in \mathbb{R}^n, F_2 \in \mathbb{R}^p, F_3 \in \mathbb{R}^p \) as:
   \[
   \begin{bmatrix}
   F_1 \\
   F_2
   \end{bmatrix}
   :=
   \begin{bmatrix}
   J_1(\dot{x}_1, u_1) - R_1(\dot{x}_1) & T(\dot{x}_1, u_1) \\
   -T^T(\dot{x}_1, u_1) & J_2(\dot{x}_1, u_1) - R_2(\dot{x}_1)
   \end{bmatrix}
   \begin{bmatrix}
   \nabla \dot{x}_1 H(\dot{x}) \\
   \nabla \dot{x}_2 H(\dot{x})
   \end{bmatrix}
   \] (4.23)
where the matrices \( J \) and \( R \) are as defined in (4.1) and
   \[
   F_3 := \nabla \dot{x}_1 \beta(\dot{x}_1) \{[J_1(\dot{x}_1, u_1) - R_1(\dot{x}_1)]\nabla \dot{x}_1 H(\dot{x}) + T(\dot{x}_1, u_1)\nabla \dot{x}_2 H(\dot{x})\}
   \] (4.23)
   \[
   -\nabla x_1 \beta(x_1) \{[J_1(x_1, u_1) - R_1(x_1)]\nabla x_1 H(x) + T(x_1, u_1)\nabla x_2 H(x)\}.
   \] (4.23)

\(^3\)A dynamical system with states \((z, y)\), input \( u \) and output \( y \), both of the same dimension, is said to be expressed in its global normal form if it is represented as
   \[
   \begin{align*}
   \dot{z} &= f_{11}(z) + f_{12}(z, y)y, \\
   \dot{y} &= f_{21}(z, y) + f_{22}(z, y)u,
   \end{align*}
   \] (4.20) (4.21)
where the square matrix \( f_{22}(z, y) \) is invertible for every \((z, y)\).
We next compute the dynamics of $\zeta_1$ and $\zeta_2$ as

\[
\begin{align*}
\dot{\zeta}_1 &= F_1(\zeta_1, \zeta_2, x_1, x_2, u_1) + L_1(\hat{x}_1, u_1)v, \\
\dot{\zeta}_2 &= \{F_2 - F_3\}(\zeta_1, \zeta_2, x_1, x_2, u_1),
\end{align*}
\tag{4.24}
\]

where we have used (4.23)-(4.23) and the fact $g_1 \equiv 0$. We note that for each, $i = 1, 2, 3$,

\[
F_i(\zeta_1, \zeta_2, x_1, x_2, u_1) = F_i(0, \zeta_2, x_1, x_2, u_1) + F_i(\zeta_1, \zeta_2, x_1, x_2 + \zeta_2, u_1),
\tag{4.25}
\]

and further $F_i(\zeta_1, \zeta_2, x_1, x_2 + \zeta_2, u_1) = 0$ whenever $\zeta_1 = 0$. So, there exists continuous matrix functions $A_i(\zeta_1, x_1, x_2 + \zeta_2, u_1) \in \mathbb{R}^{n \times n}$, $A_i(\zeta_1, x_1, x_2 + \zeta_2, u_1) \in \mathbb{R}^{p \times n}$, $i = 2, 3$, such that

\[
F_i(\zeta_1, \zeta_2, x_1, x_2 + \zeta_2, u_1) = A_i(\zeta_1, x_1, x_2 + \zeta_2, u_1)\zeta_1, i = 1, 2, 3.
\tag{4.26}
\]

We can thus see from (4.25)-(4.26) that the system (4.1), (4.24) is in its global normal form with respect to input $v$ and output $y_d(= \zeta_1)$.

Next, it is always possible to find non-negative continuous scalar functions $\psi_i(\zeta_1, x_1, x_2 + \zeta_2, u_1)$, $i = 1, 2, 3$ such that

\[
\|A_i(\zeta_1, x_1, x_2 + \zeta_2, u_1)\| \leq \psi_i(\zeta_1, x_1, x_2 + \zeta_2, u_1),
\tag{4.27}
\]

holds for all $\zeta_1, x_1, x_2 + \zeta_2, u_1$, where $\| \cdot \|$ is the induced norm of any general matrix. From the form of $J$, $R$ in (4.1) and using (4.2), (4.23) we obtain the inequality

\[
\|F_1(0, \zeta_2, x_1, x_2, u_1)\| \leq \|T(x_1, u_1)\| \|Q\zeta_2\|,
\tag{4.28}
\]

where we make use of the standard matrix norm property. We further obtain, $\|Q\zeta_2\| \leq c\sqrt{\epsilon}\|\zeta_2\|$ where $c = \bar{\lambda}_M(Q)/\sqrt{\epsilon}$ and $\epsilon$ as introduced in (4.9). We then obtain the inequalities

\[
\begin{align*}
\|\zeta_2^T Q\{F_2 - F_3\}(\zeta_1, \zeta_2, x_1, x_2 + \zeta_2, u_1)\| &\leq c\{\psi_2 + \psi_3\}(\zeta_1, x_1, x_2 + \zeta_2, u_1)\sqrt{\epsilon}\|\zeta_2\|\|\zeta_1\|, \\
\|\zeta_1^T \hat{X}L_1^{-1}(x_1 + \zeta_1)F_1(\zeta_1, \zeta_2, x_1, x_2 + \zeta_2, u_1)\| &\leq \psi_1(\zeta_1, x_1, x_2 + \zeta_2, u_1)\bar{\lambda}_M(X)\bar{\lambda}_M(L_1^{-1})\|\zeta_2\|^2, \\
\|\zeta_1^T \hat{X}L_1^{-1}(x_1 + \zeta_1)F_1(0, \zeta_2, x_1, x_2, u_1)\| &\leq c\|T(x_1, u_1)\|\bar{\lambda}_M(X)\bar{\lambda}_M(L_1^{-1})\sqrt{\epsilon}\|\zeta_2\|\|\zeta_1\|.
\end{align*}
\tag{4.29}
\tag{4.30}
\tag{4.31}
\]

We now consider the observer feedback law (4.22) with $f_1 = \psi_1\bar{\lambda}_M(\hat{X})\bar{\lambda}_M(L_1^{-1})$ and $f_2 = c\{\psi_2 + \psi_3 + \|T\|\bar{\lambda}_M(X)\bar{\lambda}_M(L_1^{-1})\}$. We differentiate the storage function $W(\zeta_1, \zeta_2) = \frac{1}{2}\zeta_2^T Q\zeta_2 + \frac{1}{2}\zeta_1^T \hat{X}\zeta_1$ along (4.1), (4.24) and use (4.29), (4.30), (4.31) to finally obtain

\[
\dot{W} \leq -\delta\|\zeta_1\|^2 + \zeta_1^Tv_d - \frac{3}{4}\epsilon\|\zeta_2\|^2 - \left\{\frac{1}{2}\sqrt{\epsilon}\|\zeta_2\| - \|\zeta_1\|f_2\right\}^2.
\]
Thus, the system is strictly passive with respect to the manifold \( M \), from input \( v_d \) to the output \( y_d = \zeta_1 \) with the storage function being \( W(\zeta_1, \zeta_2) \). Further, upon letting \( v_d = 0 \) and performing some simple computations, we get that \( \dot{W} \leq -\frac{1}{c} W \) where \( c = \max(\frac{\lambda_M(X)}{2\delta}, \frac{2\lambda_M(Q)}{3\epsilon}) \) and hence the Lyapunov function \( W(\zeta_1, \zeta_2) \) exponentially decays to zero with convergence rate \( \frac{1}{c} \).

Remark 4.11. If we let \( e = (e_1, e_2) \) denote the state estimation error, then the storage function \( W \) when expressed in the coordinates \((x, e)\) takes the form

\[
W(x, e) = \frac{1}{2} \{e_2 - \beta(x_1 + e_1) + \beta(x_1)\}^T Q \{e_2 - \beta(x_1 + e_1) + \beta(x_1)\} + \frac{1}{2} e_1^T A e_1
\]

and we thus obtain a Lyapunov function that depends both on the state and error coordinates unlike the usual quadratic error Lyapunov functions.

Remark 4.12. The inequalities (4.29), (4.30) and (4.31) are a nonlinear growth condition on \( W \) which require the growth rate to be linearly bounded in \( \zeta_2 \).

Following Remark 4.9, if \( R_2(x_1) > 0 \) then Assumption 3 holds with \( L_2 = 0 \). In this situation if we allow the matrices \( J_1, R_1 \) to also depend on \( x_2 \), then by performing some simple computations we can show that \( W \) can be linearly bounded in \( \zeta_2 \) provided the quantity \( J_1(x_1, x_2, u_1) - R_1(x_1, x_2) \) is globally Lipschitz.

A function \( f(x_1, x_2, u) \) is said to be globally Lipschitz in \( x_2 \), uniformly for all \( u_1 \in U \) if there exists a nonnegative scalar function \( \psi(x_1, u) \) such that

\[
\|f_1(x_1, x_2 + w_2, u) - f_1(x_1, x_2, u)\| \leq \psi(x_1, u)\|w_2\|. \tag{4.30}
\]

Remark 4.13. The assumption \( g_1 \equiv 0 \) ensures that (4.7) is satisfied, that is, the input \( u_2 \) is decoupled from the dynamics of \((\zeta_1, \zeta_2)\) and hence the observer design is independent of \( u_2 \). This would be the case in mechanical systems where the input is the external force applied and it appears in the dynamics of the (unmeasured) generalized momenta. When \( \beta \) is a linear function of its argument (as considered in reference [77]), the assumption \( g_1 \equiv 0 \) can be relaxed.

Remark 4.14. If \( u_1 \) is assumed to be a continuous time varying external signal taking values in a compact set \( U \subset \mathbb{R}^m \) and further has a bounded derivative, then the matrices \( L_1 \) and \( L_2 \) used in the observer dynamics can be allowed to depend smoothly on \( u_1 \). This is also natural from the view point of having to satisfy Assumption 3 because the matrix \( T \) depends on \( u_1 \). Reference [77] considers all plant inputs to be external time varying signals in their observer design.

In the next section, we illustrate our proposed observer design by considering some physical examples which come under the class of (4.1).
4. Full Order Observer Design for a class of PHSD

4.2 Physical examples

4.2.1 Permanent Magnet Synchronous Motor

We consider the permanent magnet synchronous motor example [67] with state $x = (\frac{1}{n_p} \omega, L_d i_d, L_q i_q)$, where $\omega$ is the angular velocity, $i_d, i_q$ are the currents, $j$ the moment of inertia, $n_p$ the number of pole pairs and $L_d, L_q$ the stator inductances. The Hamiltonian $H(x)$ is given as

$$H(x_1, x_2, x_3) = \frac{1}{2} n_p x_1^2 + \frac{1}{2L_d} x_2^2 + \frac{1}{2L_q} x_3^2.$$  \hspace{1cm} (4.32)

Representing the system in the rotating reference frame, we obtain the following PHSD

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix} = \begin{bmatrix} 0 & 0 & \Phi_{q_0} \\ 0 & -R_s & L_0 x_1 \\ -\Phi_{q_0} & -L_0 x_1 & -R_s \end{bmatrix} \begin{pmatrix} \nabla_{x_1} H(x) \\ \nabla_{x_2} H(x) \\ \nabla_{x_3} H(x) \end{pmatrix} + \begin{bmatrix} -\frac{1}{n_p} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} u,$$  \hspace{1cm} (4.33)

where $R_s$ is the stator winding resistance, $\Phi_{q_0}$ a constant term due to the interaction of the permanent magnet and the magnetic material in the stator and $L_0 = L_d n_p / j$. The three inputs are the the stator voltage $(v_d, v_q)$ and the constant load torque. The PHSD output is $y_p = (\omega, i_d, i_q)$, but, we assume that only the angular velocity $\omega$ is measurable, i.e $y = \omega$. It can be seen that (4.33) fits in the framework of (4.1).

We let $(\hat{x}_1, \hat{x}_2, \hat{x}_3)$ be the state estimates and define their dynamics as in (4.3). We see that the damping matrix $\begin{bmatrix} R_s & 0 \\ 0 & R_s \end{bmatrix}$ is positive definite and hence as in Remark 4.9 we choose $L_2 = 0$. Further, if $(e_1, e_2, e_3) = (\hat{x}_1 - x_1, \hat{x}_2 - x_2, \hat{x}_3 - x_3)$ denotes the estimation error, then computing the time derivative of the Lyapunov function $V(e_2, e_3) = \frac{1}{2L_d} e_2^2 + \frac{1}{2L_q} e_3^2$ along the augmented system dynamics subject to $e_1 = 0$ yields,

$$\dot{V} \leq -R_s \left\{ \frac{e_2}{L_d} \right\}^2 + \left\{ \frac{e_3}{L_q} \right\}^2,$$

$$\leq - \frac{R_s}{L_d + L_q} \left\{ e_2^2 + e_3^2 \right\},$$

$$\leq - \epsilon \left\{ e_2^2 + e_3^2 \right\},$$  \hspace{1cm} (4.34)

where $\epsilon = \frac{R_s}{(L_d + L_q)^2}$. We choose $L_1 = \frac{1}{n_p}$ and the total storage function for the plant and observer dynamics as $W(e_1, e_2, e_3) = H(e_1, e_2, e_3)$. We next compute the inequalities (4.29), (4.30), (4.31) and obtain the functions $f_1, f_2$ introduced in Theorem 4.10 as $f_1 = 0, f_2 = \frac{1}{\sqrt{e}} \frac{1}{L_q} \sqrt{L_d} \left\{ ||\hat{x}_2| + |\hat{x}_3|| \right\}$. We choose
4.2 Observers and alternate passive input-output pairs for PHSD

\[
v = -[\delta + \frac{1}{\epsilon} L_q L_d \{ |\dot{x}_2| + |\dot{x}_3| \}] e_1 + v_d \text{ and obtain the observer dynamics as}
\]

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{pmatrix} = 
\begin{bmatrix}
0 & 0 & \Phi_{q_0} \\
0 & -L_0 & L_0 \dot{x}_1 \\
-\Phi_{q_0} & -L_0 & -L_0
\end{bmatrix} 
\begin{pmatrix}
\nabla_{\dot{x}_1} H(\dot{x}) \\
\nabla_{\dot{x}_2} H(\dot{x}) \\
\nabla_{\dot{x}_3} H(\dot{x})
\end{pmatrix} + 
\begin{bmatrix}
-\frac{1}{n_p} & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} u 
- \frac{1}{n_p} [\delta + \frac{1}{\epsilon} L_q L_d \{ |\dot{x}_2| + |\dot{x}_3| \}] e_1 + \begin{bmatrix}
\frac{1}{n_p} \\
0 \\
0
\end{bmatrix} v_d.
\]

(4.35)

We finally verify that \( \dot{W} < e_1 v_d \) along (4.33), (4.35), and hence the system is strictly passive with respect to the input \( v_d \) and the output \( e_1 \).

4.2.2 Magnetic Levitation System

We consider the magnetic levitation system which was introduced in Chapter 1. It consists of an iron ball in a vertical magnetic field created by a single electromagnet, described by the model

\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{pmatrix} = 
\begin{bmatrix}
-R_2 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{bmatrix} 
\begin{pmatrix}
\nabla_{\dot{x}_1} H(x) \\
\nabla_{\dot{x}_2} H(x) \\
\nabla_{\dot{x}_3} H(x)
\end{pmatrix} + 
\begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix} u
\]

(4.36)

where \( x_1, x_2, x_3 \) correspond to the flux, position, momentum respectively and the system’s energy is given as
\( H(x_1, x_2, x_3) = \frac{1}{2m} x_3^2 + mgx_2 + \frac{1}{2k} x_1^2 \{1 - x_2\} \) with \( m \) being the mass of the ball and \( k \) is some positive constant that depends on the number of coil turns. We assume the flux and position to be measurable while the momentum cannot be measured. Thus, (4.36) fits in the framework of (4.1). We let \((\hat{x}_1, \hat{x}_2, \hat{x}_3)\) be the state estimates and define their dynamics as in (4.3). If \((e_1, e_2, e_3) = (\hat{x}_1, \hat{x}_2, \hat{x}_3) - (x_1, x_2, x_3)\) denotes the error, then upon choosing \( L_1 = \begin{bmatrix} 1 & 0 \\ 0 & m \end{bmatrix} \), \( L_2 = [0 \ m] \) in our observer construction, we obtain the zero dynamics of \((x, \dot{x})\) with respect to the outputs \((e_1, e_2)\) as \( \dot{\epsilon}_3 = -\frac{\epsilon e_3}{m} \). Then, computing the time derivative of the Lyapunov function \( V(\epsilon_3) = \frac{1}{2m} e_3^2 \) along the zero dynamics yields \( \dot{V} = -\{\frac{\epsilon e_3}{m}\}^2 \) and hence \( \epsilon = 1/\epsilon m^2 \). We introduce the change of coordinates \( \zeta = (\zeta_1, \zeta_2, \zeta_3) = (e_1, e_2, e_3 - e_2) \) to obtain the dynamics in the global normal form. We choose the total storage function as \( W(\zeta) = \frac{1}{2} \zeta_1^2 + \frac{1}{2m} \zeta_2^2 + \frac{1}{2m} \zeta_3^2 \) and compute the inequalities (4.29), (4.30), (4.31) to get \( f_1 = \frac{1}{m^2} + \frac{R_2}{k} \{1 - |\dot{x}_2| + |x_1|\} \) and \( f_2 = \frac{1}{2k} |x_1 + \dot{x}_1| \). We accordingly choose the observer feedback \( v \) as

\[
\begin{pmatrix}
v_1 \\
v_2
\end{pmatrix} = -[\delta + \frac{1}{m^2} + \frac{R_2}{k} \{1 - |\dot{x}_2| + |x_1|\} + \frac{1}{4k^2} |x_1 + \dot{x}_1|^2] \begin{bmatrix}
\zeta_1 \\
\zeta_2
\end{bmatrix} + \begin{bmatrix}
v_{d1} \\
v_{d2}
\end{bmatrix},
\]

(4.37)
and obtain the resultant observer dynamics as
\[
\begin{pmatrix}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{pmatrix} =
\begin{bmatrix}
-R_2 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0
\end{bmatrix}
\begin{pmatrix}
\nabla \dot{x}_1 H(\dot{x}) \\
\nabla \dot{x}_2 H(\dot{x}) \\
\nabla \dot{x}_3 H(\dot{x})
\end{pmatrix} +
\begin{bmatrix}
1 & 0 \\
0 & m \\
0 & m
\end{bmatrix}
\begin{bmatrix}
u_{d1} \\
v_{d2}
\end{bmatrix}
- \begin{bmatrix}
1 \\
0 \\
0
\end{bmatrix}
\begin{bmatrix}
e_1 \\
e_2
\end{bmatrix}
\left[ \delta + \frac{1}{m^2} + \frac{R_2}{k} \{ |1 - \hat{x}_2| + |x_1| \} + \frac{1}{4k^2} |x_1 + \hat{x}_1|^2 \right].
\]

We finally verify that \( W < e_1 v_{d1} + e_2 v_{d2} \) and hence the system is strictly passive with respect to the input \((v_{d1}, v_{d2})\) and the output \((e_1, e_2)\).

### 4.2.3 Inverted pendulum on a cart

We consider the inverted pendulum on a cart system which was discussed in Chapters 1 and 3. It can be modeled as

\[
\begin{pmatrix}
\dot{q} \\
\dot{\bar{p}}
\end{pmatrix} =
\begin{bmatrix}
0 & T(q) \\
-T^\top(q) & 0
\end{bmatrix}
\begin{pmatrix}
\nabla U(q) \\
\bar{p}
\end{pmatrix} +
\begin{bmatrix}
0 \\
T^\top(q) G(q)
\end{bmatrix}
u,
\]

where \( q = (q_1, q_2) \) with \( q_1 \) being the angle made by the pendulum with the vertical axis, \( q_2 \) being the horizontal position of the cart and \( \bar{p} = (\bar{p}_1, \bar{p}_2) \) are the pseudo momenta. We obtain \( \bar{p} \) by the change of coordinates \( \bar{p} = T^\top p \), where \( p = M(q) \dot{q} \) are the actual momenta with \( M(q) \) being the inertia matrix and \( TT^\top = M^{-1} \). The matrices \( T(q), G(q) \) and the potential energy function \( U(q) \) are given as

\[
T(q) =
\begin{bmatrix}
\sqrt{m_3} & 0 \\
\sqrt{m_3 - b^2 \cos^2 q_1} & 0 \\
\sqrt{m_3 - b^2 \cos^2 q_2} & 0
\end{bmatrix},
G(q) =
\begin{bmatrix}
0 \\
1
\end{bmatrix},
U(q) = a \cos(q_1).
\]

We assume that only \( q \) is measurable and see that (4.38) fits in the framework of (4.1) with \( H(q, p) = \bar{p}^\top \bar{p} + U(q) \). We next compute that \( \{ P(q) T(q) \}^\top + \{ P(q) T(q) \} > \epsilon I, \epsilon = 2 \min\{1, \frac{1}{\sqrt{m_3}}\} \) where \( P(q) = \nabla \beta(q) \) given as

\[
P(q) =
\begin{bmatrix}
b \cos q_1/m_3 & 0 \\
1 & 1
\end{bmatrix},
\beta(q) =
\begin{bmatrix}
q_1/m_3 + q_2
\end{bmatrix}.
\]

Since, \( P(q) = L_2(q) L_1^{-1}(q) \), we choose \( L_2 = P, L_1 = I_2 \times 2 \). For constructing the full order observer we introduce the coordinates \( \zeta_1 = \dot{q} - q, \zeta_2 = \dot{\bar{p}} - \bar{p} - \{ \beta(\dot{q}) - \beta(q) \} \), where \( \dot{\bar{p}} = T^\top(q) p \) and \( [\dot{q}, \dot{\bar{p}}]^\top \) is the estimate of \([q, \bar{p}]^\top\). Next,
4.2 Observers and alternate passive input-output pairs for PHSD

| \( q_1(0) = 1 \) | \( \dot{q}_1(0) = 10 \) | \( p_1(0) = 8.8513, \tilde{p}_1(0) = 67.23 \) |
| \( q_2(0) = 3 \) | \( \dot{q}_2(0) = 20 \) | \( p_2(0) = 5, \tilde{p}_2(0) = 40 \) |
| \( g = 10 \) | \( m_3 = 1 \) |
| \( b = 0.5, a = 2 \) | \( \epsilon = 5, \delta = 10 \) |

Table 4.1: Simulation parameters for the inverted pendulum example

Using standard results of functional analysis we get:

\[
\| \beta(\dot{q}) - \beta(q) \| \leq \left\{ \sup_q \| \nabla \beta(q) \| \right\} \| \zeta_1 \| ,
\]

\[
\| T^T(\dot{q}) \nabla U(\dot{q}) - T^T(q) \nabla U(q) \| \leq \left\{ \sup_q \| \nabla \{ T^T(q) \nabla U(q) \} \| \right\} \| \zeta_1 \| ,
\]

\[
\| T(\dot{q}) - T(q) \| \leq \left\{ \sup_{q_1} \| \nabla_{q_1} T \| \right\} \| \zeta_1 \| ,
\]

\[
\| P(\dot{q}) T(\dot{q}) - P(q) T(q) \| \leq \left\{ \sup_{q_1} \| \nabla_{q_1} (PT) \| \right\} \| \zeta_1 \| .
\]

If \( \lambda \) denotes the eigenvalue, then we obtain the following bounds:

\[
| \lambda(P^T P) | \leq 1 + \frac{b^2}{2m_3^2} + \frac{b}{2m_3} \sqrt{4 + \frac{b^2}{m_3^3}} = M_1,
\]

\[
| \lambda(\nabla \{ T^T \nabla U \}) | \leq a \sqrt{m_3} \sqrt{m_3 - b^2} = M_2,
\]

\[
| \lambda((\nabla_{q_1} T)^T \nabla_{q_1} T) | \leq \frac{m_3 b^2 (1 + b^2)}{(m_3 - b^2)^3} = M_3,
\]

\[
| \lambda(\nabla_{q_1} \{ PT \}) | \leq \frac{\sqrt{m_3 b^2}}{(m_3 - b^2)^2} = M_4,
\]

\[
| \lambda(T^T T) | \leq \frac{1 + m_3 + \sqrt{(1 - m_3)^2 + 4b^2}}{2(m_3 - b^2)} = M_5,
\]

\[
| \lambda(PT) | \leq \max\left\{ \frac{\sqrt{m_3}}{\sqrt{m_3 - b^2 \cos^2 q_1}}, \frac{1}{\sqrt{m_3}} \right\} = M_6.
\]

We then use the storage function \( W(\zeta_1, \zeta_2) = \frac{1}{2} \left\{ \zeta_1^T \zeta_1 + \zeta_2^T \zeta_2 \right\} \) and compute the inequalities (4.29), (4.30), (4.31) to get \( f_1 = \sqrt{M_1 M_5} + \| \tilde{p} \| \sqrt{M_3}, f_2 = \frac{1}{\sqrt{\epsilon}} \{ \sqrt{M_5} + M_2 + \| \tilde{p} \| M_4 + M_6 \sqrt{M_1} \} \). We accordingly design the observer feedback law given by (4.22) to complete the problem.

We now assume \( u = 0 \) (unforced system) and perform some simulations for the inverted pendulum on the cart example. The simulation parameters are shown in table 4.1. We also introduce additional disturbances in the measurements of \( q \) whose maximum amplitude being equal to 1% of the maximum magnitude of the measured signals during the simulation time. We present the plots showing the system and the observer trajectories with the dashed
4. Full Order Observer Design for a class of PHSD

line representing the plant state and the solid line representing the observer state. We can see that the observer is robust to the measurement disturbances and convergence is achieved.

Figure 4.1: Open-Loop trajectories for the Inverted Pendulum on Cart and the Observer with $u = 0$

Remark 4.15. As we saw in Chapter 3, the physical examples of the 3-link underactuated planar manipulator and a planar redundant manipulator with one elastic degree of freedom can also be rendered linear in the unmeasured coordinates and hence follow the same observer design methodology as the inverted pendulum on cart example. The matrices $L_1$ and $L_2$ would once again depend on the generalized position coordinate $q$ and can be computed easily for these examples.

4.2.4 Rolling Coin

We now consider the example of a coin that rolls without slipping on a plane. The dynamics of such a coin is given in [88]. If $x, y$ denote the cartesian coordinates of the point of contact of the coin with the plane, $\phi$ denotes the heading angle and $\theta$ denotes the angle of the coin’s head, then, by setting all constants to unity, the dynamics on the constrained space can be represented
4.3 Observers and alternate passive input-output pairs for PHSD

in the form

\[
\begin{pmatrix}
\dot{q} \\
\dot{\tilde{p}}
\end{pmatrix} = \begin{bmatrix}
0 & \tilde{S}(q) \\
-\tilde{S}^\top(q) & \tilde{J}(q, \tilde{p})
\end{bmatrix} \begin{pmatrix}
\nabla_q H_c(q, \tilde{p}) \\
\nabla_{\tilde{p}} H_c(q, \tilde{p})
\end{pmatrix} + \begin{pmatrix}
0 \\
B(q)
\end{pmatrix} u,
\]

(4.41)

with

\[
\tilde{S}(q) = \begin{bmatrix}
0 & \cos(\psi) \\
0 & \sin(\psi) \\
0 & 1 \\
1 & 0
\end{bmatrix},
B(q) = I_{2 \times 2}, \quad \tilde{J}(q, \tilde{p}) = 0.
\]

(4.42)

The Hamiltonian is

\[H_c(x, y, \theta, \psi, \tilde{p}_1, \tilde{p}_2) = \frac{1}{2} \tilde{p}_1^2 + \frac{1}{2} \tilde{p}_2^2\]

and we assume that only the position variables, \(q = [x, y, \theta, \psi]^T\) are measurable while the momentum variables, \(\tilde{p} = [\tilde{p}_1, \tilde{p}_2]^T\) cannot be measured. We let \(e = [\dot{q}, \dot{\tilde{p}}]^T - [q, \tilde{p}]^T\) denote the estimation error and choose the matrices \(L_1 = I_{4 \times 4}, L_2 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}\).

We then obtain the zero dynamics of the state and observer system with respect to the output \([e_1, e_2, e_3, e_4]^T\) as,

\[
\begin{align*}
\dot{e}_5 &= -e_5, \\
\dot{e}_6 &= -e_6.
\end{align*}
\]

(4.43)(4.44)

We differentiate the Lyapunov function \(V(e_5, e_6) = \frac{1}{2}e_5^2 + \frac{1}{2}e_6^2\) along the zero dynamics to obtain \(\dot{V} = -(e_5^2 + e_6^2)\) and hence \(\epsilon = 1\). We next introduce the coordinates \(\zeta = [\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5, \zeta_6]^T = [e_1, e_2, e_3, e_4, e_5 - e_4, e_6 - e_3]^T\) for expressing the system in the global normal form. We choose the total storage function as \(W(\zeta) = \frac{1}{2}\|\zeta\|^2\) and compute the inequalities (4.29), (4.30), (4.31) to obtain \(f_1 = 1 + |\cos(\psi + \zeta_4)| + |\sin(\psi + \zeta_4)| + 2|\tilde{p}_2 + \zeta_6|\) and \(f_2 = |\cos(\psi + \zeta_4)| + |\sin(\psi + \zeta_4)|\). We then accordingly choose the observer feedback law as in (4.22) which completes the problem and makes the system (4.42) and its observer strictly passive.

The potential energy of the system is zero as the coin is moving on the horizontal plane. However, the observer construction works for any arbitrary potential energy function.

4.3 A Separation Principle for PBC designs with Passivity based Observers

In this section, we consider port-Hamiltonian systems with dissipation represented as

\[
\dot{x} = [J(x_1) - R(x_1)] \nabla H(x) + g(y)u, \quad y = x_1,
\]

(4.45)
4. Full Order Observer Design for a class of PHSD

where the matrices $J$, $R$, $g$ are as defined in (4.1). This clearly belongs to the class (4.1) but with the extra property that the interconnection matrix $J$ depending only on $x_1$, which is indeed the case for all the physical examples considered in this chapter. We now prove under certain conditions, the separation principle for the system (4.45) when the proposed observer (4.3) is used in conjunction with an asymptotically stabilizing state feedback control law $u$, obtained from the well known passivity based control (PBC) design technique. We request the reader to refer to Chapters 1 and 2 for a brief discussion on PBC design.

In order to stabilize the system (4.45) at a desired equilibrium point $x = x^*$ by a PBC design technique, we aim to find a feedback law $ho(x) \in \mathbb{R}^{m \times n}$ matrices $J_d(x), R_d(x)$ depending smoothly on $x$ with $J_d^T = -J_d, R_d^T = R_d \geq 0$ and a function $H_d(x)$ having a strict minimum at $x^*$ satisfying

$$[J(x_1) - R(x_1)] \nabla H(x) + g(y) \rho(x) = [J_d(x) - R_d(x)] \nabla H_d(x).$$ (4.46)

Now upon choosing $u = \rho(x) - Kg(y) \nabla H_d(x)$ where $K^T = K > 0$ is a constant positive definite matrix, the system (4.45) in closed loop with the control action $u$ becomes,

$$\dot{x}(t) = [J_d(x) - R_d(x) - g(y)Kg^T(y)] \nabla H_d(x).$$ (4.47)

Differentiating $H_d$ along (4.47) yields

$$\dot{H}_d(x) = -(\nabla H_d(x))^T \left[ R_d(x) + g(y)Kg^T(y) \right] \nabla H_d(x),$$

$$\leq -(\nabla H_d)^T(x)g(y)Kg^T(y) \nabla H_d(x).$$ (4.49)

If the dynamics (4.47) is zero-state detectable (refer to footnote 1 in Chapter 1) with respect to the new output $\bar{y} = g^T(y) \nabla H_d(x)$, then we can establish by invoking La Salle’s invariance principle that (4.47) is asymptotically stable at the desired equilibrium point.

Usually the closed loop interconnection and damping matrices offer an extra degree of freedom for the control design and sometimes their form is a priori chosen so as to simplify the computation of the control $u$. We next make some assumptions on the closed loop interconnection, damping matrices and the closed loop energy function.

**Assumption 5.** The matrices $J_d, R_d$ depend only on the output $y = x_1$. The closed loop Hamiltonian is of the form

$$H_d(x) = \frac{1}{2} x_2^T Q_d x_2 + V_d(x_1),$$ (4.50)

where $Q_d = Q_d^T > 0$ is a constant matrix and $V_d$ has a strict minimum at $x_1^*$. Further, the function $V_d(x)$ satisfies the inequality

$$\|\nabla x_1 V_d\| \|x_1\| \leq c_3 V_d,$$ (4.51)

where $c_3$ is any positive scalar constant.
4.3 Observers and alternate passive input-output pairs for PHSD

We let \( \hat{x} = (\hat{x}_1, \hat{x}_2) \) be the estimate of \( x = (x_1, x_2) \) and introduce
\[
\Delta_u = u(\hat{x}) - u(x).
\]

Under the Assumption 5 and from (4.2), (4.46), we can see that the control \( u = \rho(x) - Kg(y)^T \nabla H_d(x) \) would be affine in the state \( x_2 \). Hence, \( \Delta_u \) satisfies the inequality,
\[
\|\Delta_u\| \leq \{\psi_0(x_1, \zeta_1) + \psi_1(x_1, \zeta_1\|x_2\|)\|\zeta_1\| + \psi_2(x_1, \zeta_1\{\|\zeta_2\| + \|\beta(\hat{x}_1) - \beta(x_1)\|\};
\]
for some nonnegative smooth functions \( \psi_0, \psi_1, \psi_2 \) and \( \zeta = (\zeta_1, \zeta_2) \) is as defined in (4.18)-(4.19). We next consider the cascaded system consisting of (4.45) with the control law \( u = \rho(\hat{x}) - Kg(y)^T \nabla H_d(\hat{x}) \), in closed loop with the observer (4.3). We then show that the trajectories of \( x_1 \) and \( x_2 \) remain bounded for all times which establishes global asymptotic stability of the cascade according to standard results from nonlinear cascaded systems theory, see [74] and the more recent work [65]. We next state the following proposition.

**Proposition 4.16.** If \( g(y) \) is assumed to be bounded and
\[
\psi_i(x_1, \zeta_1) \leq \psi_{i0}, \quad i = \{0, 1, 2\}, \tag{4.53}
\]
\[
\|\beta(\hat{x}_1) - \beta(x_1)\| \leq a\|\zeta_1\|, \quad a > 0, \tag{4.54}
\]
holds globally for some positive constants \( \psi_{00}, \psi_{10}, \psi_{20}, a \), then the port-Hamiltonian system with dissipation (4.1) with the passivity based control law \( u = \rho(\hat{x}) - Kg(y)^T \nabla H_d(\hat{x}) \) in closed loop with the observer (4.3) is globally asymptotically stable.

**Proof.** Using (4.53)-(4.54), we obtain
\[
\Delta_u \leq \{\psi_{00} + \psi_{10}\|x_2\|\}\|\zeta_1\| + \psi_{20}\{\|\zeta_2\| + \|\beta(\hat{x}_1) - \beta(x_1)\|\},
\]
\[
\leq \psi_{10}\|\zeta\||x| + \psi_{20}\|\zeta_2\| + \{\psi_{00} + a\psi_{20}\}\|\zeta_1\|,
\]
\[
\leq \psi_{10}\|\zeta\||x| + \bar{\psi}_{20}\|\zeta\|, \tag{4.55}
\]
where \( \bar{\psi}_{20} = \psi_{00} + (1 + a)\psi_{20} \). We next employ the feedback law \( u = \rho(\hat{x}) - Kg(y)^T \nabla H_d(\hat{x}) \) in (4.45) to obtain
\[
\dot{x} = [J_d(x) - R_d(x) - g(y)Kg^T(y)]\nabla H_d(x) + g(y)\Delta_u. \tag{4.56}
\]
We subsequently compute the derivative of \( H_d(x) \) along (4.56) and use (4.55) to obtain the inequality,
\[
\dot{H_d} \leq -k'\|g^T(y)\nabla H_d(x)\|^2 + \|g^T(y)\nabla H_d(x)\{\psi_{10}\|\zeta\||x| + \bar{\psi}_{20}\|\zeta\|\}, \tag{4.57}
\]
where \(k' = \tilde{\lambda}_{\text{min}}(K) > 0\). Next, by invoking standard young’s inequality argument (refer to footnote 1 of Chapter 3) we get
\[
\|g^T(y) \nabla H_d(x)\| \|\zeta\| \leq \frac{k'}{\psi_{20}} \|g^T(y) \nabla H_d(x)\|^2 + \frac{\psi_{20}}{4k'} \|\zeta\|^2.
\] (4.58)

Using (4.58) in (4.57) we get
\[
\dot{H}_d \leq \frac{\psi_{20}^2}{4k'} \|\zeta\|^2 + \psi_{10} \|g^T(y) \nabla H_d(x)\| \|\zeta\| \|x\|,
\] (4.59)
\[
\leq \frac{\psi_{20}^2}{4k'} \|\zeta\|^2 + \psi_{10} \|g(y)\| \|\nabla H_d(x)\| \|\zeta\| \|x\|,
\] (4.60)
\[
\leq \frac{\psi_{20}^2}{4k'} \|\zeta\|^2 + c' \psi_{10} \|g(y)\| \|\zeta\| H_d(x),
\] (4.61)

where we have used (4.51) to obtain the second term in the inequality (4.61) with \(c' > 0\) being some positive constant. We recall from the proof of Theorem 4.10 that the storage function \(W(\zeta_1, \zeta_2)\) satisfies the inequality \(\dot{W} \leq -\delta' W\). Further, since \(W\) is quadratic in \(\zeta\) we can always find a positive constant \(\gamma\) such that \(\gamma \dot{W} \leq -\frac{\psi_{20}^2}{4k'} \|\zeta\|^2\). Consequently, by choosing \(V_1 = H_d + \gamma W\) and differentiating it along the system dynamics, we would obtain
\[
\dot{V}_1 \leq c' \psi_{10} \|g(y)\| \|\zeta\| H_d(x),
\]
\[
\leq c' \psi_{10} \|g(y)\| \|\zeta\| V_1,
\]
\[
\leq c' \psi_{10} \psi_3 \|\zeta\| V_1,
\]

where we assume from the boundedness of \(g(y)\) that \(\|g(y)\| \leq \psi_3\). Finally, by solving the differential inequality we obtain
\[
V_1(t) \leq V_1(0)e^{\int_0^t c' \psi_{10} \psi_3 \|\zeta\| d\tau},
\] (4.62)

which implies that \(V_1(t)\) is bounded for all \(t \geq 0\), for any finite initial error \(\zeta(0)\). Subsequently, we get from the continuity and positive definiteness of \(V_1\) that the trajectories of \(x\) are bounded for all \(t \geq 0\). We can now conclude that the system (4.1) with control input \(u(\hat{x})\) in closed loop with the observer (4.3) is global asymptotic stable and hence the result follows. \(\square\)

### 4.4 Numerical example

In this section we verify our theoretical results by performing some simulations for the magnetic levitation example (4.36). We show that, when the observer variables are used in place of state variables in the passivity based control law, the closed loop system still reaches the desired equilibrium point, \((\sqrt{2mgk}, x^*_2, 0)\).
4.4 Observers and alternate passive input-output pairs for PHSD

<table>
<thead>
<tr>
<th>$x(0) = (5, 0.7, 3)$</th>
<th>$\hat{x} = (3, 0.1, 5)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g = 10$</td>
<td>$m = 1$</td>
</tr>
<tr>
<td>$K_p = 0.75, \alpha = 2$</td>
<td>$x_2^* = 0.5, k = 5$</td>
</tr>
<tr>
<td>$R_a = 5, R_2 = 5$</td>
<td>$\delta = 5$</td>
</tr>
</tbody>
</table>

Table 4.2: Simulation parameters for the inverted pendulum example

The passivity based controller for the system (4.36) has been studied in [69] and is given as

$$u = \frac{R_2 x_1}{k} \{1 - x_2\} - \{\frac{\alpha}{m} + K_p R_a\} x_3 - K_p \{\frac{1}{\alpha} \tilde{x}_1 + \tilde{x}_2\},$$

(4.63)

where $R_a > 0$, $K_p, \alpha$ are constants and $\tilde{x}_1 = \{x_1 - \sqrt{2mgk}\}$, $\tilde{x}_2 = \{x_2 - x_2^*\}$. Employing this feedback law leads to the closed loop plant dynamics

$$\left(\begin{array}{c}
\dot{x}_1 \\
\dot{x}_2 \\
\dot{x}_3
\end{array}\right) = \left[
\begin{array}{ccc}
0 & 0 & -\alpha \\
0 & -R_a & 1 \\
\alpha & -1 & 0
\end{array}\right]
\left(\begin{array}{c}
\nabla x_1 H_d(x) \\
\nabla x_2 H_d(x) \\
\nabla x_3 H_d(x)
\end{array}\right),$$

(4.64)

where the closed loop Hamiltonian is given as

$$H_d = \left\{\frac{1}{2 m} + \frac{K_p R_a}{2 \alpha}\right\} x_3^2 - \frac{m g \tilde{x}_1}{\alpha} + \frac{x_1^3}{6 k \alpha} + \frac{K_p x_3}{\alpha} \{\frac{1}{\alpha} \tilde{x}_1 + \tilde{x}_2\} + \frac{K_p}{2 R_a \alpha} \{\frac{1}{\alpha} \tilde{x}_1 + \tilde{x}_2\}^2.$$  

(4.65)

It can be verified that the closed loop Hamiltonian in (4.65) has a minimum at the desired equilibrium point, $(\sqrt{2mgk}, x_2^*, 0)$. We next note that the feedback law (4.63) is affine in the unmeasured momenta $x_3$ and hence the arguments presented in the previous section on the separation principle would hold for this example. We now use the state estimates $\hat{x}_1, \hat{x}_2, \hat{x}_3$ whose dynamics satisfy (4.38) in the feedback law (4.63) to obtain the closed-loop system consisting of the plant, controller and the observer.

The simulation parameters are shown in Table 4.2. In order to testify the robustness of the observer-controller design, we introduce disturbances in the measurements of flux and position whose maximum amplitude being equal to 1% of the maximum magnitude of the measured signals during the simulation time. Figure 4.2 shows the unforced system trajectories (solid lines) of the flux, position and momentum along with their estimates (dashed lines). We can see that the observer is robust in the presence of disturbance and converges to the plant states. Figure 4.3 compares the plots of the system trajectories obtained by using the output feedback control and the normal full state feedback control. We can see that the system reaches the desired equilibrium point $(10, 0.5, 0)$ with output feedback (in the presence of disturbances), thus showing the efficacy of the proposed observer design.
4. Full Order Observer Design for a class of PHSD

Figure 4.2: State (solid line) and observer (dashed line) trajectories for the unforced system

4.5 Conclusion

We have proposed a passivity based full-order observer design framework for a class of port-Hamiltonian systems with dissipation which leads to the construction of a globally exponentially stable observer. The idea is to render the augmented system (composed of the plant and the observer dynamics) strictly passive with respect to an invariant manifold defined on the extended state space on which the state estimation error is zero. We also obtained as a part of the full-order observer construction, a globally exponentially stable reduced-order observer.

The observer construction is done in two steps:
(1) Compute the observer gain matrices $L_1(\hat{x}_1)$ and $L_2(\hat{x}_1)$ by solving a set of algebraic and partial differential equations such that the augmented system has a vector relative degree \{1, ...., 1\} and is globally minimum phase with respect to the manifold $M$ (defined in (4.5)).
(2) Compute the partial state feedback law, $v(y, \hat{x}, u_1)$ by following the constructive procedure given in the proof of Theorem 4.10, in order to render the augmented system strictly passive with respect to the manifold $M$.

Without providing an algorithmic procedure for finding $L_1(\hat{x}_1)$ and $L_2(\hat{x}_1)$ we have instead shown in a couple of well-known physical examples how to compute them. In some of these examples the observer design is performed using constant $L_1$, $L_2$ matrices, but by considering the classical example of the inverted pendulum on the cart we have shown that it is natural to allow $L_1$, $L_2$ to depend on $\hat{x}_1$.

Finally, we have proved by invoking concepts from nonlinear cascaded sys-
4.5 Observers and alternate passive input-output pairs for PHSD

Figure 4.3: System trajectories shown using state (solid line) and output feedback (dashed line)

Systems theory that the separation principle holds when the proposed observer is used in closed loop with an asymptotically stabilizing state feedback control law that is obtained by a passivity based control (PBC) design approach.
4. Full Order Observer Design for a class of PHSD