Multi-model unfalsified switching control of uncertain multivariable systems

S. Baldi 1, G. Battistelli 2, D. Mari 2,*,†, E. Mosca 2 and P. Tesi 3

1 Computer Science Department, University of Cyprus (UCY), 1678 Nicosia, Cyprus
2 Dipartimento Sistemi e Informatica, DSI - Università di Firenze, Via S. Marta 3, 50139 Firenze, Italy
3 Department of Communications, Computer and System Sciences, DIST - University of Genoa, Via Opera Pia 13, 16145 Genova, Italy

SUMMARY

This paper addresses the problem of controlling an uncertain multi-input multi-output system by means of adaptive switching control schemes. In particular, the paper aims at extending the multi-model unfalsified control approach, so far restricted to single-input single-output systems, to a general multivariable setting. The proposed scheme relies on a data-driven 'high-level' unit, called the supervisor, which at any time can switch on in feedback with the uncertain plant one controller from a finite family of candidate controllers. The supervisor performs routing and scheduling tasks by monitoring suitable test functionals which, on the basis of the measured data, provide a measure of mismatch between the potential loop made up by the uncertain plant in feedback with the candidate controller and the nominal 'reference loop' related to the same candidate controller. Copyright © 2012 John Wiley & Sons, Ltd.

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1. INTRODUCTION

In many control systems, such as industrial plants, aircraft, and communication networks, a large number of actuator and/or sensors may be employed in order to achieve the desired control task. In these kind of control problems, the inputs and outputs cannot usually be grouped into pairs and treated as if they were separate single-input single-output (SISO) problems, because the interactions between the multiple inputs and outputs are non-negligible. In this case, one has to tackle control design as a genuine multiple-input multi-output (MIMO) problem. The problem is even more complicated when the multivariable system to be controlled is poorly known. One of the approaches for controlling uncertain plants is the introduction of feedback adaptation. The extension of adaptive control algorithms developed for SISO systems to a MIMO setting is not trivial. Some MIMO adaptive control algorithms based on the model reference approach and the pole placement approach can be found in [1, 2]. In recent years, adaptive switching control (ASC) (see Figure 1) has emerged as an alternative to conventional continuous adaptation, providing an attractive framework for combining tools from adaptive and robust control [3–7]. ASC usually embeds a multicontroller consisting of a family of precomputed candidate controllers and a supervisor that orchestrates the switching by selecting at any time a specific controller among the candidate controllers family, based on plant

*Correspondence to: D. Mari, Dipartimento Sistemi e Informatica, DSI - Università di Firenze, Via S. Marta 3, 50139 Firenze, Italy.
†E-mail: mari@dsi.unifi.it

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input/output data [8]. Although the literature on ASC is quite vast, most of the works deal with the SISO case with notable exceptions being [9–12].

This paper aims at extending the multi-model unfalsified ASC (MMUASC) approach of [7, 13] to a MIMO setting. In MMUASC, the supervisor performs in real-time the scheduling task (when to switch) and the routing task (which controller select), by monitoring suitable test functionals, pairwise associated with the given candidate controllers, as indicators of controller suitability. Each test functional provides a measure of percentage discrepancy between the potential loop, made up by the uncertain plant in feedback with the candidate controller, and a nominal ‘reference loop’ related to the same candidate controller. While, in the SISO case, such a discrepancy can be obtained by resorting to the concept of a virtual reference, the situation becomes more intricate in a MIMO setting because, in this case, such virtual references need not exist. It will be constructively shown that, irrespective of the existence of such virtual references, the proposed approach still maintains intuitive interpretations and enjoys the same robustness features as the original one.

The paper is organized as follows. Section 2 provides an overview of the problem. Section 3 introduces the MMUASC approach and discusses the main question arising in the MIMO case. Sections 4 and 5 sum up the main features of the adopted test functionals. In particular, the stability properties of the resulting closed-loop system are analyzed in Section 5. It is also shown how the basic scheme can be suitably modified so as to ensure, along with stability, asymptotic tracking properties. Section 6 discusses a simulation example. Section 7 ends the paper with concluding remarks.

2. SWITCHING CONTROL FRAMEWORK

Let the ‘switched’ system be represented as follows:

\[
\begin{align*}
\dot{y}(t) &= \mathcal{P}(u)(t) \\
u(t) &= \mathcal{C}_{\sigma(t)}(r - y)(t)
\end{align*}
\]

where \( t \in \mathbb{Z}_+ \), \( \mathbb{Z}_+ := \{0, 1, \cdots\} \), \( \mathcal{P} : u \mapsto y \) denotes the uncertain plant with input \( u(t) \in \mathbb{R}^m \) and output \( y(t) \in \mathbb{R}^p \); \( r(t) \in \mathbb{R}^p \) denotes the reference to be tracked by the plant output and \( \sigma(t) \) the subscript identifying the candidate controller connected in feedback to the plant at time \( t \).

It is assumed that the uncertain plant consists of a discrete-time strictly causal MIMO Linear Time-Invariant (LTI) dynamic system with matrix fraction descriptions (MFDs, for short)

\[
\mathcal{P} : A^{-1}(d)B(d) = N(d)D^{-1}(d),
\]

where \( A(d) = I_p + A_1 d + \cdots + A_n d^n \) and \( B(d) = B_1 d + \cdots + B_m d^m \) are polynomial matrices in the unit backward-shift operator \( d \) with strictly Schur greatest common left divisor (g.c.l.d.). Similar definitions apply to the right MFD \( N(d)D^{-1}(d) \).

A supervisory unit handles the plant I/O records in order to generate the sequence \( \sigma \) specifying the switching controller \( \mathcal{C}_{\sigma(t)} \). Specifically, at each time \( t \), the controller \( \mathcal{C}_{\sigma(t)} \) is one element from
a finite family \( \mathcal{C} = \{C_i, \ i \in \hat{N} \} \), of one-degrees-of-freedom LTI controllers \( C_i \) with MFDs

\[
C_i : R_i^{-1} (d) \ S_i (d) = Y_i (d) \ X_i^{-1} (d),
\]

where \( R_i (d) = I_n + \mathcal{B}_1 d + \cdots + \mathcal{B}_n d^n \) and \( S_i (d) = \mathcal{A}_0 + \mathcal{A}_1 d + \cdots + \mathcal{A}_n d^n \) are polynomial matrices with strictly Schur g.c.l.d. As beforehand, a similar definition applies to the right MFDs \( Y_i (d) \ X_i^{-1} (d) \). The control action \( u(t) \) is realized via shared state multicontroller implementations [8] so as to both reduce the switching transients and make it possible the use of unstable controllers.

An example of a common state multicontroller consists of using as a shared state the vector

\[
\varepsilon(t) := \left[ e(t-1)', \cdots, e(t-q)', u(t-1)', \cdots, u(t-q) \right]',
\]

where \( e := r - y, q := \max \{ \deg S_i, \deg R_i, \ i \in \hat{N} \} \), with \( \deg M \) standing for to the highest degree of all the entries of \( M(d) \) and setting for the output of \( C_\sigma(t) \) at \( t \)

\[
C_\sigma(t) (e)(t) := [\mathcal{A}_\sigma(t) \cdots \mathcal{A}_\sigma(t)_q - \mathcal{A}_\sigma(t)_1 \cdots - \mathcal{A}_\sigma(t)_0] e(t) + \mathcal{A}_\sigma(t)_0 e(t).
\]

The linear time-varying feedback system (1) will be denoted by \((\mathcal{P}/\mathcal{C}_\sigma(t))\). Let now \( \mathcal{S} \) denote the linear space of all the real-valued sequences on \( \mathbb{Z}_+ \). Then, given a vector-valued sequence \( x \in \mathcal{S} \) of dimension \( n \), \( x^t \) denotes its time truncation up to time \( t \), that is, \( x^t := \{x(0), x(1), \ldots, x(t)\} \), with \( x(k) \in \mathbb{R}^n \). Finally, \( \|x^t\|^2 := \sum_{k=0}^{t} |x(k)|^2 \) where \( | \cdot | \) denotes Euclidean norm.

**Definition 1**

The switched system (1) is said to be stable relatively to \( r \) (\( r \)-stable, for short) if for every input \( r \in \mathcal{S} \), there exist finite positive reals \( c_i, \ i = 1, 2 \), such that

\[
\|z^t\| := c_1 + c_2 \|r^t\|, \quad t \in \mathbb{Z}_+,
\]

where \( z(k) := [u(k)' y(k)']' \).

Let \( \mathcal{P} \) represent the set of all possible plant transfer matrices. At times, we shall write

\[
\mathcal{P} := \{\mathcal{P}(\theta), \ \theta \in \Theta\},
\]

and denote by \( \theta \) the vector of parameters composed by the coefficients of \( A_i(d) \) and \( B_i(d) \), taking values in some set \( \Theta \). In order for the problem to be well-posed, the following requirement is assumed.

**Definition 2**

The ASC problem is said to be feasible if, for every \( \mathcal{P} \in \mathcal{P} \), there is at least an index \( i \in \hat{N} \) such that \((\mathcal{P}/\mathcal{C}_i)\) is internally stable.

Before proceeding, some comments are in order. For clarity of exposition, in the remainder of this paper, the analysis will be carried out assuming zero plant initial conditions and zero noises/disturbances. Nonetheless, the results to be presented can be readily extended to the general case along the same lines as those of [7,14]. In accordance with the mentioned restrictions, next definition is introduced in order to avoid possible ambiguities.

**Definition 3**

Given an LTI dynamic system with transfer matrix \( F(d) \), and left MFD, \( F(d) = G^{-1}(d) H(d) \), with input \( u \) and output \( y \), by the notation \( y(t) = F(d) u(t) \), we mean that the sequence \( y(t), \) \( t \in \mathbb{Z}_+ \), is computed via the following difference equation (\( \det G_0 \neq 0 \))
\[
\sum_{k=0}^{n_G} G_k y(t-k) = \sum_{k=0}^{n_H} H_k u(t-k), \quad y(k) = u(k) = 0, \quad k = -1, -2, \cdots ,
\]

where \( G(d) = \sum_{k=0}^{n_G} G_k d^k \) and \( H(d) = \sum_{k=0}^{n_H} H_k d^k \).

In order to decide which candidate controller has to be placed in feedback with plant, the supervisor embodies a family \( \Pi := \{ \Pi_i, i \in \hat{N} \} \) of test functionals such that, in broad terms, \( \Pi_i(t) \) quantifies the suitability of the \( i \)-th \emph{potential loop} \((\mathcal{P}/\mathcal{C}_i)\) given the data up to time \( t \). In the hysteresis switching logic (HSL), considered hereafter, at each step, one computes the least index \( i_*(t) \) in \( \hat{N} \) such that \( \Pi_{i_*(t)}(t) \leq \Pi_i(t), \forall i \in \hat{N} \). Then, the switching index sequence \( \sigma \) is given by

\[
\begin{align*}
\sigma(t+1) &= l(\sigma(t), \Pi(t)), \quad \sigma(0) = i_0 \in \hat{N} \\
l(i, \Pi(t)) &= \begin{cases} 
    i, & \text{if } \Pi_i(t) < \Pi_{i_*(t)}(t) + h \\
    i_*(t), & \text{otherwise}
\end{cases}
\end{align*}
\]

where \( h > 0 \) is the hysteresis constant.

The next lemma establishes the limiting behavior of \((\mathcal{P}/\mathcal{C}_{\sigma(t)})\) subject to (8). Let \( \Sigma \) denote the class of all possible switching sequences \( \sigma \) giving rise to the switched system (1). Consider the assumptions

**A1.** For each \( \sigma(\cdot) \in \Sigma \) and \( i \in \hat{N} \), \( \Pi_i(t) \) admits a limit (even infinite) as \( t \to \infty \).

**A2.** For each \( \sigma(\cdot) \in \Sigma \), there exist integers \( \mu \in \hat{N} \) such that \( \Pi_{i_*(t)}(t) \) is bounded.

**Lemma 1**

**HSL Lemma** [3]. Let \( z \) denote the vector-valued sequence of I/O plant data and \( \sigma(\cdot) \) the switching sequence resulting from (1) and (8). Then, for any initial condition and reference \( r \), if A1 and A2 hold, there is a finite time \( t_f \in \mathbb{Z}_+ \), after which no more switching occurs. Moreover, \( \Pi_{\sigma(\cdot)}(t) \) is bounded.

3. REFERENCE-LOOP IDENTIFICATION IN THE MULTIVARIABLE CASE

As discussed in [7], the combination of multiple models architectures and Unfalsified Control has appealing features and intuitive advantages. Indeed, the resulting approach, called MMUASC (multi-model unfalsified adaptive switching control), was shown to combine the positive features of both approaches in that it reduces the time duration of learning transients and stability is ensured if for any element in the plant uncertainty set, there is at least one stabilizing candidate controller. In this section, we briefly review the main features of the MMUASC approach and discuss a number of questions related to the multivariable case.

Let \( \mathcal{M} := \{ \mathcal{M}_i, i \in \hat{N} \} \) be a family of \( N \) discrete-time strictly causal MIMO LTI dynamic systems with MFDs

\[
\mathcal{M}_i : \quad A_i^{-1}(d) B_i(d) = N_i(d) D_i^{-1}(d),
\]

where \( A_i(d) = I_p + A_{i1}d + \cdots + A_{i_n}d^n \) and \( B_i(d) = B_{i1}d + \cdots + B_{in_b}d^n \) are polynomial matrices with strictly Schur greatest g.c.l.d. Similar definitions apply to the right MFDs \( N_i(d) D_i^{-1}(d) \). \( \mathcal{M} \) is taken as a representative set of all possible plant transfer matrices and form, along with \( \mathcal{F} \), a finite family \( \mathcal{F} = \{ (\mathcal{M}_i/\mathcal{C}_i), i \in \hat{N} \} \) of internally stable feedback loops, each designed to fulfill desired prescriptions. Hereafter, \((\mathcal{M}_i/\mathcal{C}_i)\) will be referred to as the \( i \)-th reference-loop or nominal-loop.
In MMUASC, the aim is to carry out a *reference-loop identification* task, viz., select a candidate controller $C_i$ in such a way that $(P/C_i)$ behaves as closest as possible to one of the candidate reference loops in $F$. The underlying idea can be described as follows: At time $t$, we would like to update the controller index in (8) on the basis of the test functionals

$$
\Pi_i(t) = \max_{r \in \mathcal{R}} \frac{\|[(P/C_i)r - (M_i/C_i)r]^T\|}{\|[(M_i/C_i)r]^T\|},
$$

where $(P/C_i)r$ and $(M_i/C_i)r$ denote the behavioral data produced by $(P/C_i)$ and $(M_i/C_i)$, respectively, in response to $r$. The maximum operator means that we search for the controller minimizing the discrepancy between actual and nominal behavior with respect to the whole observation interval up to the current time $t$.

**Remark 1**

As elaborated in [15, 16], the reason for using percentage criteria is essentially that, in case of large uncertain plant dynamic range, we can have a different cost associated to each index $i \in \mathcal{N}$. Test functionals in normalized form like (10) thus help to avoid possible biases associated with the controller selection.

The test functional (10) would allow us to compare the performance levels achievable by the use of each candidate controller, $(P/C_i)$ being $r$-stable (not falsified by $(r,z)$ using the terminology of [6]) if and only if $\Pi_i(\cdot)$ is bounded. Unfortunately, online computation of (10) is impossible without using logics like pre-routing, which in general have to be ruled out because it typically cause large and long-lasting learning transients. In fact, online computation of (10) would require to compute the response of $(P/C_i)$ to $r$ for each candidate controller, which is not possible unless all the controllers are sequentially tested. The unfalsified control approach introduced in [17] provides, under certain conditions, a way to side-step this problem. At each time and for each candidate controller, one computes (if possible) the solution $v_i(t)$ of the difference equation

$$
S_i(d)v_i(t) = R_i(d)u(t) + S_i(d)y(t).
$$

In words, $v'_i$ is equals the *virtual reference sequence* which would reproduce the recorded I/O sequence $z'$ should the plant $P$ be fed back by the candidate controller $C_i$, irrespective of the way $z'$ is generated. This means that, if $(P/C_{a(t_i)})$ denotes the linear (time-varying) transformation (1) mapping the $r$ into $z$, we have $z = (P/C_{a(t_i)})r = (P/C_i)v_i$.

In MMUASC, the virtual reference concept is used as follows. For each reference-loop $(M_i/C_i)$, we define the closed-loop response of $(M_i/C_i)$ to $u_i$ as

$$
y_{i/i}(t) = M_i(u_{i/i}(t))
$$

and

$$
u_{i/i}(t) = C_i(u_i - y_{i/i}(t)).
$$

Accordingly, by letting $(M_i/C_i) v_i := [u'_{i/i} y'_{i/i}]^T$ (Figure 2), the test functionals (10) are modified as follows:

Figure 2. Detail of a MMUASC scheme.
where $G$.

Consider the linear equation

$$ Gx = L, \quad (14) $$

where $G \in \mathbb{R}^{m \times p}$ and $L \in \mathbb{R}^{m}$ are given matrices. Then, the following statements are equivalent:

(i) There exists a solution $x \in \mathbb{R}^{p}$.

(ii) The columns of $L \in \text{Im} G$.

Furthermore, the solution, if it exists, is unique if and only if $G$ has full column rank.

**Lemma 3**

Consider the linear equation (14), where $m \leq p$ and $G$ has full row rank, that is, rank $G = m$. Then, the solution, if any, can be expressed as

$$ x = G^\dagger L + (I_p - G^\dagger G)\gamma, \quad (15) $$

where $\gamma \in \mathbb{R}^{p}$, whereas $G^\dagger := (G'G')^{-1}$ denotes right pseudo-inverse of $G$.

Let us rewrite (11) as follows

$$ \mathcal{J}_{i0}v_i(t) = R_i(d)u(t) + S_i(d)y(t) + [\mathcal{J}_{i0} - S_i(d)]v_1(t) := \xi_i(t). \quad (16) $$

On the basis of the aforementioned lemmas, the following conclusions can be drawn regarding the solution of (11) in the multivariable case.

(1) $m = p$. If $\mathcal{J}_{i0}$ has full column rank, the virtual reference $v_i(t) = \mathcal{J}_{i0}^{-1} \xi_i(t)$ always exists unique. Moreover, if $S_i(d)$ is strictly Schur, then equation (11) is numerically stable.

(2) $m < p$. If $\mathcal{J}_{i0}$ has full row rank, $(\mathcal{J}_{i0}^\dagger \mathcal{J}_{i0})$ is invertible and all the possible $v_i$'s are given by

$$ v_i(t) = \mathcal{J}_{i0}^\dagger \xi_i(t) + (I_p - \mathcal{J}_{i0}^\dagger \mathcal{J}_{i0})v(t), \quad (17) $$

where $v \in \mathbb{R}^{p}$ is an arbitrary signal. Moreover, if $S_i(d)$ is strictly Schur, then equation (11) is numerically stable.

(3) $m > p$. The virtual reference $v_i$ need not exist unless $\xi_i(t) \in \text{Im} \mathcal{J}_{i0}$.

Case (3) motivates the adoption of a virtual reference different from the one in (11). To this end, let $w(t) := S_{\sigma(t)}(d)r(t)$. An equivalent representation of system (1) is depicted in Figure 3 where the polynomial matrices $R_{\sigma(t)}(d)$ and $S_{\sigma(t)}(d)$ are now in the forward path and respectively, in the backward path of control loop. Accordingly, we can consider, in place of $v_i$, the signal

$$ w_i(t) = R_i(d)u(t) + S_i(d)y(t) \quad (18) $$

so that the $i$-th virtual control loop of Figure 2 becomes the one depicted in Figure 4. In the light of (18) and Figure 4, we can therefore replace the test functionals in (13) by

$$ \Pi_i(t) = \max_{t \leq t} \frac{\left\| \left[ (P/C_i)w_i - (M_i/C_i)w_i \right]^T \right\|}{\left\| \left[ (M_i/C_i)w_i \right]^T \right\|}, \quad (19) $$

3.1. Virtual references for multivariable systems

In the SISO case [7], the virtual references are always well-defined, and questions arise only on their numerical computation (the solution of (11) being ‘numerically stable’ only if $S_i(d)$ is strictly Schur, that is, only if $C_i$ is minimum-phase). The introduction for MIMO systems of a test functional like (13) is instead a more intricate matter, because the virtual reference in (11) need not even exist, as we show. To this end, some results of linear algebra are briefly recalled [18].

**Lemma 2**

Consider the linear equation

$$ Gx = L, \quad (13) $$

where $G$.

Furthermore, the solution, if it exists, is unique if and only if $G$ has full column rank.
where, with obvious meaning of the symbols, \(( \mathcal{P}/\mathcal{C}_i )\) and \(( \mathcal{M}_i/\mathcal{C}_i )\) denote the \(i\)-th potential loop and the \(i\)-th reference loop, respectively.

In contrast with (13), the test functional (19) is always well-defined, because (18) is such. Moreover, computation of \(w_i\) is always numerically stable\(^2\). This leads us to make the following remarks:

From a conceptual point of view, the MMUASC approach based on (19) still maintains an interpretation in terms of discrepancy between potential and nominal loops, both driven by a virtual reference signal. From a practical point of view, even when \(v_i\) is well-defined, the MMUASC approach based on (19) does not pose questions related to numerical aspects. This latter issue is addressed in more detail in the next section, where implementation aspects are discussed.

### 4. STABILITY INFERENCE AND IMPLEMENTATION ASPECTS

For simplicity, let us rewrite (19) as follows:

\[
\Pi_i(t) := \max_{t \leq T} \Lambda_i(t) ,
\]

\[
\Lambda^{1/2}_i(t) := \frac{\| \bar{z}_i \|}{\| (\bar{z} - \bar{z}_i)' \|} ,
\]

with \(\bar{z}_i := z - z_i, z = (\mathcal{P}/\mathcal{C}_i)w_i\) and \(z_i = (\mathcal{M}_i/\mathcal{C}_i)w_i := [u_i' \ y_i']', i \in \bar{N}\).

We now discuss the main features of (19) in terms of stability inference and implementation. Consider the co-prime factor uncertainty on \(\mathcal{P}\) based on \(\mathcal{M}_i\),

\[
\Delta_{*i}(d) := \begin{bmatrix} \Delta B_i(d) & \Delta A_i(d) \end{bmatrix} = \begin{bmatrix} B(d) - B_i(d) & A_i(d) - A(d) \end{bmatrix} ,
\]

and let

\[
\Xi_{i/i}(d) := A_i(d)X_i(d) + B_i(d)Y_i(d) ,
\]

\[
\Xi_{*i}(d) := A(d)X_i(d) + B(d)Y_i(d) ,
\]

whose determinants is equal to the characteristic polynomials of the \(i\)-th reference loop and respectively, the \(i\)-th potential loop.

---

\(^2\)For SISO systems, test functionals based on \(w_i\) were indeed considered in order to obtain numerically stable solutions in the presence of non-minimum phase controllers [19].
The next lemma establishes the main features of (19) in terms of stability inference.

**Lemma 4**
Consider the polynomial matrices

\[
Q_{i/i}(d) := \begin{bmatrix} Y_i(d) \\ X_i(d) \end{bmatrix} \Xi_{i/i}^{-1}(d),
\]

\[
Q_{\ast/i}(d) := \begin{bmatrix} Y_i(d) \\ X_i(d) \end{bmatrix} \Xi_{\ast/i}^{-1}(d).
\]

Then,

\[
Q_{\ast/i}(d) - Q_{i/i}(d) = Q_{\ast/i}(d) \Delta_{i/i}(d) Q_{i/i}(d).
\]

**Proof**
See the appendix.

From Lemma 4, it is immediate to conclude that

\[
\tilde{z}_i(t) = Q_{i/i}(d) \Delta_{i/i}(d) z(t)
\]

\[
= Q_{\ast/i}(d) \Delta_{\ast/i}(d) \left[ I - Q_{i/i}(d) \Delta_{i/i}(d) \right] z(t)
\]

\[
= Q_{\ast/i}(d) \Delta_{\ast/i}(d) (z(t) - \tilde{z}_i(t)).
\]

It follows from (28) that the boundedness of (21) only depends on the stability of \((\mathcal{P}/\mathcal{C}_i)\). In fact, for the switched-on controller, say \(\mathcal{C}_i\), the corresponding test functional in (21) can grow unbounded if and only if \(\mathcal{C}_i\) does not stabilize the plant (this property being known as cost-detectability \([6]\)). As shown next, this is exactly the property which ensures \(r\)-stability of the switched system.

Before proceeding to prove the stability, we discuss the main implementation aspects related to (21). The sequence \(\tilde{z}_i\) can be obtained via \(\tilde{z}_i = z - \tilde{z}_i\), where \(z_i\) can be computed by running, as shown in Figure 4, with \(\mathcal{P}\) replaced by \(\mathcal{M}_i\), the reference-loop \((\mathcal{M}_i/\mathcal{C}_i)\) driven by \(w_i\). As noted previously, such a procedure does not pose any question related to numerical aspects. Nonetheless, it is convenient to provide an alternative way for computing \(\tilde{z}_i\).

**Lemma 5**
Consider the vector valued sequence \(\tilde{z}_i\) in (21). Then,

\[
\tilde{z}_i(t) = \begin{bmatrix} -Y_i(d) \\ X_i(d) \end{bmatrix} \Xi_{i/i}^{-1}(d) \epsilon_i(t),
\]

where \(\epsilon_i(t) := A_i(d) y(t) - B_i(d) u(t)\) is the prediction error based on \(\mathcal{M}_i\).

**Proof**
See the appendix.

For SISO systems \([7]\), (29) was suggested as an alternative procedure for computing (13) in the presence of non-minimum phase controllers, \(\text{viz.}\) in case the computation of the \(v_i\)'s were not numerically stable. Hence, from a practical point of view, the MMUASC approach based on (19) can be seen as the direct counterpart of the original MMUASC approach for SISO systems. The merit of (19) is to recover, even for multivariable systems, an interpretation of the test functional in terms of discrepancy between potential and nominal loops, which is a form of identification for control.

We finally observe that, for both the procedures outlined earlier, the resulting test functionals (21) are only used in order to update the controller switching index, with no need to implement the switching controller as in Figure 3.
5. MAIN RESULTS

In this section, the stability properties of the switched system are analyzed. In this respect, we observe that (28) would be sufficient per se to prove $r$-stability of the switched system. Nonetheless, in order to fully exploit the advantage of using a multiple model architecture, some preliminary observations are needed.

Given any $\mathcal{P} \in \mathcal{P}$, let $S(\mathcal{P}) \subseteq \overline{\mathcal{N}}$ be the set of all indices $s \in \overline{\mathcal{N}}$ such that $(\mathcal{P}/\mathcal{C}_s)$ is stable. Obviously, under problem feasibility, $S(\mathcal{P}) \neq \emptyset$ holds for any $\mathcal{P} \in \mathcal{P}$. Whenever the plant uncertainty set $\mathcal{P}$ is compact and a priori known, $\mathcal{M}$ can be designed dense enough in $\mathcal{P}$ so as to ensure that, for any $\mathcal{P} \in \mathcal{P}$, there exist indices $i \in \overline{\mathcal{N}}$, yielding stable loops $(\mathcal{P}/\mathcal{C}_i)$ and such that $\|Q_{s/i} \Delta_{s/i}\|_\infty < \beta$, where $\| \cdot \|_\infty$ stands for the $\mathcal{H}_\infty$-norm and $\beta$ is a design parameter. More specifically, the following result can be stated, whose proof follows along the same lines of the SISO case (see [13]).

**Proposition 1**

Let $\Theta$ be a compact set and $\theta \mapsto \mathcal{P}(\theta)$ continuous on $\Theta$. Then, for any positive real $\beta$, there always exists a finite model family such that

$$\max_{\mathcal{P} \in \mathcal{P}} \min_{i \in S(\mathcal{P})} \|Q_{s/i} \Delta_{s/i}\|_\infty : = \beta < \beta.$$ (30)

A model distribution for which property (30) holds will be denoted by $\mathcal{M}(\beta)$. The main result of this section can be stated.

**Theorem 1**

Consider the switched system (1) under zero initial conditions. Let $\sigma(\cdot)$ be selected in accordance with the HSL (8), with test functionals as in (20)–(21). Then, under problem feasibility, for any reference $r \in S$, the switched system is $r$-stable. Further, under a model distribution $\mathcal{M}(\beta)$, the total number of switches $N_\sigma$ is bounded as follows:

$$N_\sigma \leq N \left\lceil \frac{\beta^2}{h} \right\rceil,$$ (31)

where $\lceil \alpha \rceil$ denotes the smallest integer greater than or equal to $\alpha \in \mathbb{R}_+$.

**Proof**

See the appendix. \qed

**Remark 2**

It can be shown that Theorem 1 also extends to the cases of nonzero plant initial conditions and/or nonzero disturbances. As for the first issue, see [7]. In relation to the latter issue, a detailed discussion on how to ensure cost-detectability in noisy environments can be found in [14].

5.1. Tracking properties

Hereafter, we discuss how the previous results can be extended so as to ensure asymptotic tracking. We assume that the reference $r(t) \in \mathbb{R}^p$ is such that

$$\Phi(d)r(t) = 0,$$ (32)

where $\Phi(d) := \phi(d) I_p$, $\phi(0) = 1$ and whose elements have only simple roots on the unit circle. This amounts to assuming that $r(t) i$ is a bounded periodic sequence.

Consider a left MFD of the plant (1), as specified by (2). Necessary and sufficient conditions for the existence of a linear time-invariant controller ensuring asymptotic tracking are the following [20]:
(a) $m \geq p$.
(b) The polynomial matrices $B(d)$ and $\Phi(d)$ are left co-prime.

In the present case, (b) is understood to hold for every $P \in \mathcal{P}$. Multiplying (32) by $A(d)$ and (2) by $\Phi(d)$ and subtracting the resulting equations, we obtain

$$A(d) \Phi(d) e(t) = B(d) \eta(t).$$

where $e(t) = r(t) - y(t)$ and $\eta(t) := \Phi(d) u(t)$. In such a case, any stabilizing controller $C_i$ of the form

$$\Phi(d) u(t) = Y_i(d) X_i^{-1}(d) e(t),$$

ensures asymptotic tracking. Such a controller has transfer matrix $\Phi^{-1}(d) Y_i(d) X_i^{-1}(d)$ and in agreement with the so called Internal Model Principle, which incorporates the model of the reference to be tracked [20–22].

With the use of (33) and (34), the switched system can be therefore rewritten as

$$e(t) = \bar{P}(\eta(t))$$

$$\eta(t) = C_{\sigma(t)}(-e)(t) \right \}

$$

where $\bar{P}$ denotes the ‘new’ plant with input $\eta$, output $e$, and transfer matrix as in (33). Essentially, this means that the tracking problem for system (1) is transformed into an equivalent zero-regulation problem for (35).

By exploiting such an equivalence, a simple approach for ensuring, along with stability, the offset-free tracking property, consists in designing the family $\mathcal{R}$ of nominal loops $(M_i / C_i)$’s in such a way that

(c) For each candidate model $M_i$, the polynomial matrices $B_i(d)$ and $\Phi(d)$ are left co-prime.
(d) Each candidate controller $C_i$ stabilizes the corresponding model $M_i$ and ensures offset-free tracking in the sense of (34).

Under such design conditions, we can modify the test functional (21) by replacing $z$ with $\zeta = [\eta' e']'$ and similarly, $z_i$ with $\zeta_i$. This leads to

$$L_i^{1/2}(t) := \frac{\| \tilde{\zeta}_i \|}{\| (\zeta - \tilde{\zeta}_i)^t \|},$$

where $\tilde{\zeta}_i := \zeta - \zeta_i$. The so-modified test functional now provides a measure of discrepancy between potential and nominal loops with respect to system (35), by which we get at once the following.

**Theorem 2**

Consider the switched system (1) under zero initial conditions. Let $\sigma(\cdot)$ be selected in accordance with the HSL (8), with test functionals as in (20)–(36). Let $r(t)$ satisfy (32) and assume that conditions (a)–(b) hold. Further, assume that $\mathcal{R}$ has been designed so as to satisfy conditions (c)–(d). Then, under problem feasibility, the switched system is $r$-stable and offset-free.

**Proof**

See the appendix.

6. **EXAMPLE**

Consider the plant $P$ of Figure 5, made up by four carts mechanically coupled by springs and dampers, where the control problem consists in positioning the external carts by applying manipulable forces to the internal ones. The resulting plant is a two-input and two-output systems.

The carts have a mass of $m = 1$ Kg and the dampers have a viscous damping coefficient $c$ equal to 0.1 Ns/m. Let now $\theta = [\theta^{[1]} \theta^{[2]} \theta^{[3]}]' \in \mathbb{R}^3$, the vector denotes the value of stiffness of the
three springs. In all simulations reported hereafter, the hysteresis constant \( h \) was set equal to 0.05 and reference signals \( r_1(t) \) and \( r_2(t) \) which are square-waves of amplitudes and periods of \( \pm 2.5 \) m and \( 150 \) s and \( \pm 1.5 \) m and \( 100 \) s, respectively. Further, the controller index \( \sigma \) is selected in accordance with the HSL (8) with test functionals (20)–(21), assuming zero plant initial condition and zero noises and disturbances.

Consider first the case of a monodimensional uncertainty where only the spring connecting the carts on the left has an uncertain stiffness parameter \( \theta^{[1]} \in \Theta = [0.1, 1.75] \) N/m, whereas the other springs are assumed to have a known stiffness coefficient \( \theta^{[2]} = \theta^{[3]} = 0.7 \) N/m. For this case, three different one-degree-of-freedom continuous LTI controllers were designed in order to guarantee the stability and performance requirements on the whole uncertain interval \( \Theta \). The controllers were designed relatively to plant models corresponding to the following three stiffness values: \( \theta^{[1]}_1 = 0.2 \) N/m, \( \theta^{[1]}_2 = 0.5 \) N/m and \( \theta^{[1]}_3 = 1.0 \) N/m. Specifically, let \( M_i(s) = A_i^{-1}(s)B_i(s), \quad i \in \hat{\Delta} \) denote the plant model with stiffness \( \theta^{[1]}_i \) from \( u = [u_1 \ u_2]' \) to \( y = [y_1 \ y_2]' \) and \( C_i(s) = Y_i(s)X_i(s)^{-1} \) the corresponding controller, which was selected among all stabilizing controllers \( \hat{C}_i(s) = \hat{Y}(s)\hat{X}^{-1}(s) \) in accordance with the following weighted \( H_\infty \) mixed-sensitivity criterion [23]

\[
C_i(s) = \arg \inf_{\hat{C}} \sup_\omega \| \Phi_W(j\omega, \theta_i) \|,
\]  

(37)

where \( \bar{\sigma} \) denotes the maximum singular value and \( \Phi_W(s, \theta_i) \) denotes the \( W \)-weighted mixed sensitivity matrix

\[
\Phi_W(s, \theta_i) = \begin{bmatrix}
\{\Psi_{ij}^{[1]}\}_{ij=1}^{-1/2}W_i(s)\hat{Y}(s) \\
\{\Psi_{ij}^{[2]}\}_{ij=1}^{-1/2}W_i(s)\hat{X}(s)
\end{bmatrix}\bar{\sigma}^{-1}_i(s)A_i(s),
\]  

(38)

Figure 5. Four carts plant.

Figure 6. Stability range in case of monodimensional uncertainty and three controllers \( \Theta^{[1]}_1 = (0.1, 0.46), \Theta^{[1]}_2 = (0.26, 0.96), \) and \( \Theta^{[1]}_3 = (0.72, 1.75) \) N/m.
with \( \tilde{Z}_i(s) := A_i(s) \tilde{X}(s) + B_i(s) \tilde{Y}(s) \). The weighting polynomial matrices and the positive real-valued matrices have been chosen as follows: \( W_i(s) = 0.01/(s + 0.01)I_2 \), \( \Psi_i^y = I_2 \), \( i \in \{3\} \), and \( \Psi_i^u = 2.85 \times 10^{-3} I_2 \), \( \Psi_i^w = 2.14 \times 10^{-3} I_2 \), \( \Psi_i^u = 1.00 \times 10^{-3} I_2 \). Then, nominal models and relative controllers have been discretized by means of an input zero-order holder with sampling time equal to 0.1 s. Figure 6 shows the stability subintervals \( \Theta_{[1]} \) of each controller.

Table I and Figure 7 report experiments relative to different stiffness values of the uncertain spring and different initial controllers. In particular, Figure 7 also depicts the time-behavior of the test functionals related to the candidate controllers.

We next consider a case of bidimensional uncertainty, which requires a larger number of controllers. Specifically, we assume that both the external springs have the following uncertain

<table>
<thead>
<tr>
<th>( \theta[1] )</th>
<th>Initial controller index</th>
<th>Final controller index</th>
<th>Final switching time (s)</th>
<th>Maximum values of</th>
<th>Maximum values of</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.15</td>
<td>3</td>
<td>1</td>
<td>0.70</td>
<td>2.62 and 1.71</td>
<td>4.91 and 3.55</td>
</tr>
<tr>
<td>0.25</td>
<td>2</td>
<td>1</td>
<td>1.20</td>
<td>2.54 and 1.61</td>
<td>4.91 and 3.55</td>
</tr>
<tr>
<td>0.35</td>
<td>3</td>
<td>2</td>
<td>0.80</td>
<td>2.59 and 1.70</td>
<td>4.58 and 3.64</td>
</tr>
<tr>
<td>0.40</td>
<td>1</td>
<td>2</td>
<td>1.30</td>
<td>2.54 and 1.64</td>
<td>4.59 and 3.67</td>
</tr>
<tr>
<td>0.65</td>
<td>3</td>
<td>2</td>
<td>2.35</td>
<td>2.51 and 1.61</td>
<td>4.59 and 3.68</td>
</tr>
<tr>
<td>0.75</td>
<td>1</td>
<td>3</td>
<td>4.20</td>
<td>2.71 and 1.77</td>
<td>8.88 and 4.68</td>
</tr>
<tr>
<td>0.90</td>
<td>1</td>
<td>3</td>
<td>0.90</td>
<td>2.50 and 1.54</td>
<td>5.46 and 4.69</td>
</tr>
<tr>
<td>1.20</td>
<td>2</td>
<td>3</td>
<td>0.90</td>
<td>2.49 and 1.54</td>
<td>5.46 and 4.69</td>
</tr>
<tr>
<td>1.70</td>
<td>1</td>
<td>3</td>
<td>0.80</td>
<td>2.51 and 1.56</td>
<td>9.96 and 4.68</td>
</tr>
</tbody>
</table>

Figure 7. Tracking, control action, controller selection, and test functionals in case of monodimensional uncertainty and three controllers. Left, \( \theta[1] = 0.30 \) and \( \sigma(0) = 3 \); Right, \( \theta[1] = 0.80 \) and \( \sigma(0) = 1 \).
Figure 8. Left: stability range in case of bidimensional uncertainty and nine controllers. Controllers are obtained by setting $W_1(s) = 0.01(s + 0.01)I_2$, $\Psi_1^f = I_2$ and $\Psi_i^m = 10^{-3}I_2$, $i = 1, 2, \ldots, 9$. Right: values corresponding to the nominal models.

Table II. Simulation results in case of bidimensional uncertainty and nine controllers.

| Initial controller index | Final controller index | Final switching time (s) | Maximum values of $|y_1|$ and $|y_2|$ | Maximum values of $|u_1|$ and $|u_2|$ |
|------------------------|-----------------------|-------------------------|--------------------------------|-----------------------------------|
| $[0.25, 0.30]$         | 6                     | 1                       | 0.90                           | 2.72 and 1.71                     | 7.86 and 5.81                     |
| $[0.30, 1.30]$         | 8                     | 3                       | 0.70                           | 2.51 and 1.59                     | 7.95 and 4.50                     |
| $[0.60, 0.20]$         | 2                     | 4                       | 0.80                           | 2.51 and 1.55                     | 8.87 and 5.83                     |
| $[0.40, 1.20]$         | 1                     | 6                       | 0.80                           | 2.66 and 1.69                     | 6.45 and 4.36                     |
| $[0.70, 0.70]$         | 5                     | 8                       | 2.70                           | 2.64 and 1.73                     | 5.39 and 5.03                     |
| $[1.00, 0.65]$         | 9                     | 8                       | 34.10                          | 3.41 and 2.70                     | 5.40 and 11.77                    |
| $[0.90, 1.40]$         | 1                     | 9                       | 0.70                           | 2.56 and 1.53                     | 10.37 and 4.34                    |
| $[1.25, 1.30]$         | 3                     | 9                       | 0.70                           | 2.53 and 1.51                     | 14.43 and 4.34                    |
| $[1.40, 0.60]$         | 4                     | 8                       | 0.80                           | 2.51 and 1.51                     | 9.44 and 4.98                     |
| $[0.35, 0.40]$         | 7                     | 2                       | 32.70                          | 3.87 and 2.98                     | 8.21 and 11.72                    |
| $[0.30, 0.85]$         | 8                     | 5                       | 0.80                           | 2.54 and 1.68                     | 7.87 and 4.42                     |
| $[0.50, 0.90]$         | 9                     | 6                       | 0.90                           | 2.53 and 1.52                     | 6.46 and 4.33                     |
| $[1.40, 0.30]$         | 3                     | 7                       | 0.70                           | 2.60 and 1.51                     | 14.44 and 5.87                    |
| $[0.55, 0.55]$         | 1                     | 5                       | 0.80                           | 2.52 and 1.54                     | 9.52 and 4.99                     |
| $[0.80, 0.35]$         | 1                     | 5                       | 37.10                          | 3.57 and 2.44                     | 9.64 and 4.99                     |

stiffness: $\theta^{[1]} \in [0.18, 1.6]$ N/m and $\theta^{[3]} \in [0.18, 1.6]$ N/m, whereas the internal spring takes on a known constant value $\theta^{[2]} = 0.7$ N/m. Accordingly, the uncertainty set is given by $\Theta = [0.18, 1.6] \times [0.18, 1.6]$ N/m. The values corresponding to nine nominal models are indicated in Figure 8, along with the stability ranges of their corresponding controllers. We used the $H_\infty$ mixed-sensitivity criterion of (37)–(38) in order to obtain the candidate controllers. Table II shows a set of simulation results for different values of the uncertainty and different initial controllers, whereas Figure 9 depicts two of the most critical scenarios, as reported in Table II. Consistent with intuition, the most critical cases are those involving plant parameters close to the boundary regions of destabilizing controllers. Nonetheless, from the performance point of view, the closed-loop behavior always remains satisfactory, the plant input/output always being kept at a moderate level. Finally, Tables III and IV show the variation of $\beta$ in (30) as a function of $N$, with the plant models logarithmically distributed over the uncertainty set. This appears as a convenient choice in view of the fact that the plant is harder to control for smaller values of the spring stiffness.
Figure 9. Tracking, control action, and controller selection in case of bidimensional uncertainty and nine controllers. Left, \([\theta_1, \theta_3] = [0.35, 0.40]\) and \(\sigma(0) = 7\); Right, \([\theta_1, \theta_3] = [0.80, 0.35]\) and \(\sigma(0) = 1\).

<table>
<thead>
<tr>
<th>(N)</th>
<th>3</th>
<th>6</th>
<th>9</th>
<th>12</th>
<th>15</th>
<th>18</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\bar{\beta})</td>
<td>7.17</td>
<td>1.78</td>
<td>1.16</td>
<td>0.94</td>
<td>0.79</td>
<td>0.69</td>
<td>0.3</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>(N)</th>
<th>9</th>
<th>16</th>
<th>25</th>
<th>36</th>
<th>49</th>
<th>64</th>
<th>81</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\bar{\beta})</td>
<td>20.84</td>
<td>23.72</td>
<td>10.93</td>
<td>8.80</td>
<td>7.43</td>
<td>5.36</td>
<td>4.56</td>
</tr>
</tbody>
</table>

### 7. CONCLUSIONS

In this paper, we addressed the problem of controlling uncertain systems by means of switching control techniques. Specifically, we described how the MMUASC approach introduced in [7] extends to the case of a MIMO plant. By resorting to a different notion of virtual references, the same stability and performance features of the original MMUASC carry over to the multivariable case with no additional assumptions on the plant to be controlled. We also described a simple variant of the basic scheme which ensures, along with stability, the offset-free tracking property. The effectiveness of the proposed scheme has been discussed and illustrated through a numerical example.
Throughout the Appendix, we make use of the following result.

**Lemma 6**
Let $\Psi_{i/i} := R_i D_i + S_i N_i$, whose determinant is equals to that of (23). Then, the following relationships hold

\[ Y_i \Xi_{i/i}^{-1} A_i = D_i \Psi_{i/i}^{-1} S_i, \tag{39} \]
\[ X_i \Xi_{i/i}^{-1} B_i = N_i \Psi_{i/i}^{-1} R_i. \tag{40} \]

**Proof of Lemma 6.**
Consider first (39) and notice that

\[ S_i A_i^{-1} \Xi_{i/i} = S_i A_i^{-1} (A_i X_i + B_i Y_i) \]
\[ = S_i X_i + S_i N_i D_i^{-1} Y_i \]
\[ = R_i Y_i + (\Psi_{i/i} - R_i D_i) D_i^{-1} Y_i \]
\[ = \Psi_{i/i} D_i^{-1} Y_i, \]

where the third equality follows from $S_i X_i = R_i Y_i$ and the definition of $\Psi_{i/i}$. In turns,

\[ S_i A_i^{-1} \Xi_{i/i} = \Psi_{i/i} D_i^{-1} Y_i \Rightarrow \Psi_{i/i}^{-1} S_i A_i^{-1} = D_i^{-1} Y_i \Xi_{i/i}^{-1} \Rightarrow D_i \Psi_{i/i}^{-1} S_i = Y_i \Xi_{i/i}^{-1} A_i. \]

Likewise,

\[ \Xi_{i/i} X_i^{-1} N_i = (A_i X_i + B_i Y_i) X_i^{-1} N_i \]
\[ = A_i N_i + B_i R_i^{-1} S_i N_i \]
\[ = B_i D_i + B_i R_i^{-1} (\Psi_{i/i} - R_i D_i) \]
\[ = B_i R_i^{-1} \Psi_{i/i} \],

where the third equality follows from $B_i D_i = A_i N_i$ and the definition of $\Psi_{i/i}$. Hence,

\[ \Xi_{i/i} X_i^{-1} N_i = B_i R_i^{-1} \Psi_{i/i} \Rightarrow X_i^{-1} N_i \Psi_{i/i} = \Xi_{i/i}^{-1} B_i R_i^{-1} \Rightarrow N_i \Psi_{i/i}^{-1} R_i = X_i \Xi_{i/i}^{-1} B_i. \]

\[ \square \]

**Proof of Lemma 4.**
It follows from

\[ \Xi_{*|i}^{-1} - \Xi_{|i}^{-1} = \Xi_{*|i}^{-1} (I_p - \Xi_{*|i} \Xi_{|i}^{-1}) \]
\[ = \Xi_{*|i}^{-1} (\Xi_{i/i} - \Xi_{*|i}) \Xi_{|i}^{-1} \]
\[ = \Xi_{*|i}^{-1} \begin{bmatrix} \Delta B_i & \Delta A_i \end{bmatrix} \begin{bmatrix} -Y_i \\ X_i \end{bmatrix} \Xi_{|i}^{-1}. \]

\[ \square \]

**Proof of Lemma 5.**
Because $u_i$ and $y_i$ in (21) are obtained by

\[ u_i = D_i \Psi_{i/i}^{-1} w_i, \quad y_i = N_i \Psi_{i/i}^{-1} w_i, \]
by using (39) and (40), it follows that
\[ u - u_i = u - D_i \Psi_i^{-1} w_i \]
\[ = u - D_i \Psi_i^{-1} (R_i u + S_i y) \]
\[ = (I_m - D_i \Psi_i^{-1} R_i) u - D_i \Psi_i^{-1} S_i y \]
\[ = (D_i - D_i \Psi_i^{-1} R_i D_i) D_i^{-1} u - Y_i \Xi_i^{-1} A_i y \]
\[ = (D_i - D_i \Psi_i^{-1} (\Psi_i^j - S_i N_i)) D_i^{-1} u - Y_i \Xi_i^{-1} A_i y \]
\[ = D_i \Psi_i^{-1} S_i A_i^{-1} B_i u - Y_i \Xi_i^{-1} A_i y \]
\[ = Y_i \Xi_i^{-1} B_i u - Y_i \Xi_i^{-1} A_i y \]
\[ = -Y_i \Xi_i^{-1} (A_i y - B_i u) , \]
and with similar algebra, \( y - y_i = X_i \Xi_i^{-1} (A_i y - B_i u) . \)

Proof of Theorem 1.
From (28), the transfer matrix of the system mapping \( z - \bar{z}_i \) to \( \bar{z}_i \) coincides with \( Q_{*i/j} \Delta_{*i/j} . \) Therefore, under feasibility, Assumption \( \textbf{A2} \) is satisfied. Moreover, also, Assumption \( \textbf{A1} \) is satisfied because of the maximum operator. Thus, Lemma 1 holds, and controller switching always stops in a finite time for every reference sequence \( r \in \mathcal{S} . \) By Lemma 1, the test functional \( \Lambda_f^{1/2} \) related to the final switched-on controller \( C_f \) is bounded, \( \forall i, \) there exists a positive real \( \kappa \) such that
\[ \Lambda_f^{1/2}(t) \leq \kappa , \quad \forall t \in \mathbb{Z}_+ . \]
Then, by triangular inequality, one has
\[ \| z_f \| \leq \kappa \| (z - \bar{z}_f) \| + \| \bar{z}_f \| \leq (1 + \kappa) \| (z - \bar{z}_f) \| . \]
Recalling (28), one finds that
\[ z - \bar{z}_f = z - Q_{f/f} \Delta_{*f/f} z(t) \]
\[ = z - \left[ \begin{array}{c} -Y_f \\ X_f \end{array} \right] \Xi_f^{-1} f (A_f y - B_f u) \]
\[ = z - \left[ \begin{array}{c} -Y_f \\ X_f \end{array} \right] \Xi_f^{-1} f A_f (y - N_f D_f^{-1} u) . \]
(41)
The first \( m \) rows of (41) yield
\[ u + Y_f \Xi_f^{-1} f A_f (y - N_f D_f^{-1} u) = D_f \Psi_f^{-1} S_f y + (I_m - D_f \Psi_f^{-1} S_f N_f D_f^{-1}) u \]
\[ = D_f \Psi_f^{-1} S_f y + [I_m - D_f \Psi_f^{-1} (\Psi_f/R_f D_f) D_f^{-1}] u \]
\[ = D_f \Psi_f^{-1} S_f y + D_f \Psi_f^{-1} R_f u \]
\[ = D_f \Psi_f^{-1} \left[ \begin{array}{cc} R_f & S_f \end{array} \right] z , \]
(42)
where the first inequality follows from (39). Likewise, the last \( p \) rows of (41) yield
\[ y - X_f \Xi_f^{-1} f A_f (y - N_f D_f^{-1} u) = (I_p - X_f \Xi_f^{-1} f A_f) y + X_f \Xi_f^{-1} f B_f u , \]
as \( A_f N_f = B_f D_f . \) From (40), we get \( X_f \Xi_f^{-1} f B_f = N_f \Psi_f^{-1} f R_f . \) Moreover,
\[ I_p - X_f \Xi_f^{-1} f A_f = X_f \Xi_f^{-1} f A_f \left( A_f^{-1} \Xi_f^{-1} f X_f^{-1} - I_p \right) \]
\[ = X_f \Xi_f^{-1} f A_f \left[ A_f^{-1} (A_f X_f + B_f Y_f) X_f^{-1} - I_p \right] \]
\[ = X_f \Xi_f^{-1} f B_f (Y_f X_f^{-1}) \]
\[ = N_f \Psi_f^{-1} f R_f (R_f^{-1} S_f) . \]
which finally yields
\[ y - X_f \Xi^{-1}_{f f} A_f (y - N_f D_f^{-1} u) = N_f \Psi^{-1}_{f f} \begin{bmatrix} R_f & S_f \end{bmatrix} z. \] (43)

Combining (42) and (43), (41) can be therefore rewritten as
\[ z - \tilde{z}_f = \begin{bmatrix} D_f \\ N_f \end{bmatrix} \Psi^{-1}_{f f} \begin{bmatrix} R_f & S_f \end{bmatrix} z. \]

Further, regardless of the state the controller $C_f$ is in at time $t^*$, one has $R_f u(t) + S_f y(t) = S_f r(t)$ for $t > t^* + q_f$ with $q_f = \max\{\deg S_f, \deg R_f\}$. This, along with the fact that $\Psi_{f f}$ is strictly Schur, implies that
\[ \|z - \tilde{z}_f\| \leq \alpha \|r^f\| + \beta \]
for some positive reals $\alpha$ and $\beta$. Hence, $r$-stability follows at once.

**Proof of Theorem 2.**
Exploiting the results of Theorem 1, we obtain that switching stops onto some candidate controller $C_f$ and $\Lambda^1_{f f}(t) \leq k_1$ for some positive real $k_2$. By virtue of the design conditions (c) and (d), $\xi_f(\cdot)$ converges to zero and hence, $\|\xi^f_{f}\| \leq k_2$ for some positive real $k_2$. By triangular inequality, we obtain
\[ \|\xi^f\| \leq k_1 (1 + k_2) =: \kappa \] (44)
from which we conclude that $\xi(\cdot)$ converges to zero. This proves the offset-free tracking property.

As for $r$-stability, notice first that (44) implies
\begin{enumerate}
  \item $\|y^f\| \leq \kappa + \|r^f\|$;
  \item $\|y^f\| \leq \kappa$.
\end{enumerate}

By Bezout identity, (b) implies the existence of two polynomial matrices $U$ and $V$ such that
\[ \Phi U + V B = I \] (45)

Multiplying both the sides of (45) by $u$, we get $U u + V A y = u$. By combining the last equality with (i) and (ii), we obtain the desired result.

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**REFERENCES**


