Multi-model unfalsified adaptive switching control
Baldi, S.; Battistelli, G.; Mosca, E.; Tesi, Pietro

Published in: International Journal of Adaptive Control and Signal Processing

DOI: 10.1002/acs.1225

IMPORTANT NOTE: You are advised to consult the publisher’s version (publisher’s PDF) if you wish to cite from it. Please check the document version below.

Document Version
Publisher's PDF, also known as Version of record

Publication date: 2011

Link to publication in University of Groningen/UMCG research database

Citation for published version (APA):

Copyright
Other than for strictly personal use, it is not permitted to download or to forward/distribute the text or part of it without the consent of the author(s) and/or copyright holder(s), unless the work is under an open content license (like Creative Commons).

Take-down policy
If you believe that this document breaches copyright please contact us providing details, and we will remove access to the work immediately and investigate your claim.

Downloaded from the University of Groningen/UMCG research database (Pure): http://www.rug.nl/research/portal. For technical reasons the number of authors shown on this cover page is limited to 10 maximum.
Multi-model unfalsified adaptive switching control: Test functionals for stability and performance

S. Baldi, G. Battistelli, E. Mosca and P. Tesi*

Dipartimento Sistemi e Informatica, DSI—Università di Firenze, Via S. Marta 3, 50139 Firenze, Italy

SUMMARY

The paper studies how to infer behavioral features of a potential loop, consisting of an uncertain plant interconnected in feedback with a candidate controller, on the grounds of experimental data taken from the same plant possibly driven by a different controller. In such a context, convenient tools to work with are test functionals, computed via a virtual experiment, which quantify the discrepancy between the potential loop and the so-called ‘tuned-loop’ or ‘reference-loop’ related to the same candidate controller. Several test functionals are considered and analyzed so as to unveil conditions under which their adoption can accomplish the desired goals. These results are shown to be of practical relevance for on-line performance inference of feedback control systems and implementation of highly performing adaptive switching control systems. Copyright © 2011 John Wiley & Sons, Ltd.

Received 9 October 2009; Revised 13 October 2010; Accepted 17 November 2010

KEY WORDS: control of uncertain plants; adaptive control; switching supervisory control

1. INTRODUCTION

The main goal of this paper is to reconsider, from a wider viewpoint, the choices adopted in [1] concerning the multi-model approach to unfalsified adaptive switching control (MMUASC). In fact, Reference [1] proposes and analyses a specific MMUASC scheme based on a test functional with no comparison with alternative conceivable test functionals. In other words, while Reference [1] shows that a particular MMUASC system built via the adoption of a specific test functional enjoys several desirable properties, the present paper considers various possible MMUASC systems built via the adoption of different conceivable test functionals, and aims at comparing their respective properties in terms of generality of applicability, simplicity of implementation, and performance.

To this end, the basic scheme of a generic adaptive switching control (ASC) system is first described (Figure 1). In ASC, a data-driven ‘high-level’ unit $\mathcal{S}$, called the supervisor, aims at controlling an uncertain plant $\Pi$, belonging to an uncertainty set $\mathcal{P}$, by switching at any time in feedback with $\Pi$ one controller from a finite family $\mathcal{K} = \{K_i, i \in \mathbb{N}\}$, $\mathbb{N} := \{1, 2, \ldots, N\}$, of causal candidate controllers. In the simplest nontrivial case considered throughout the paper, $\mathcal{P}$ consists of a set of discrete-time strictly causal finite-dimensional SISO LTI dynamic systems. The controller selection is carried out by $\mathcal{S}$ based on the plant input $\dot{u}$ and output $y$. The supervisor performs in real-time the scheduling task (when to switch) and the routing task (which controller to switch-on), by monitoring suitable test functionals, pairwise associated with the given candidate controllers, as indicators of controller performance.

*Correspondence to: P. Tesi, Dipartimento Sistemi e Informatica, DSI—Università di Firenze, Via S. Marta 3, 50139 Firenze, Italy.
†E-mail: ptesi@dsi.unifi.it

Copyright © 2011 John Wiley & Sons, Ltd.
The search for suitable test functionals is a crucial task underlying many control application areas aimed at testing and improving closed-loop performance, e.g., on-line controller design in the framework of identification for control [2–5], the performance assessment of feedback control systems [6–9]. As will be seen, this is in particular the case for ASC systems considered in the present paper. In fact, in ASC, the schemes that appear as the most promising are those where the supervisor jointly performs routing and scheduling, i.e., the schemes in which the supervisor evaluates on-line the potential performance of each candidate controller and selects the controller corresponding to the best inferred performance.‡ In this connection, special attention has to be paid to the test functionals choice, since inadequate choices facilitate temporary insertions in the loop of destabilizing controllers, and, hence, cause undesirable transients.

Related approaches to ASC are usually classified as either indirect (also known as Multi-model ASC (MMASC) schemes) [13–15] or direct (also known as Unfalsified ASC (UASC) schemes) [16–18]. In MMASC, the test functionals are selected so as to provide a plant-model error estimate, which is usually expressed via $\ell_2$-norms of sequences of prediction errors based on a family $M := \{M_i, i \in \mathbb{N}\}$ of ‘representative’ nominal models, each model associated with a related candidate controller. In such a way, the supervisor selects the candidate controller whose related nominal model best approximates the uncertain plant. It is known that, in MMASC, transients can be typically made small if the plant uncertainty set $P$ is tightly approximated by $M$. However, it also known that if the latter condition is not enforced, neither convergence to a final controller nor boundedness are in general ensured [13]. This situation looks antithetic in UASC, where the test functionals are selected so as to directly forecast closed-loop performance achievable by using each candidate controller, with no intermediate identification effort. UASC schemes in fact ensure finite-time convergence for switching and stability in-the-large under the only assumption that a stabilizing candidate controller exist. Nevertheless, they typically yield significant transients before the final controller be switched-on [1].

A recent paper [1] considered a multi-model approach in the UASC context. The scheme proposed in [1] showed that it is possible to combine the positive features of indirect and direct approaches in the following sense: by reducing, as in MMASC, the time duration of learning transients after start-up in case $P$ be tightly approximated by a nominal model distribution $M$; by also preserving, as in UASC, stability irrespective of $M$, viz. irrespective of possible plant-model mismatches, thus allowing application in the case of large plant uncertainties.

As anticipated, the main goal of this paper is to compare various possible MMUASC systems, built via the adoption of different conceivable test functionals, from the viewpoint of their generality of applicability, simplicity of implementation, and performance. We also provide, for each MMUASC system analyzed, sufficient conditions on $P$ under which the families $K$ and $M$ can be constructed so as to ensure stability of the resulting switched system.

The paper is organized as follows: Section 2 sums up the foundations of UASC systems and points out merits and shortcomings of related approaches wherein no plant models are used.

‡For this reason, we will not address here pre-routing-based schemes or variants thereof [10–12].
Section 3 introduces MMUASC, discusses how to select appropriate model-based test functionals for stability and performance inference, and provides several comments and comparisons among the various test functionals. Section 4 describes relevant implementational issues and extends the conclusions to a noisy environment. Section 5 concludes the paper by summing up results and open problems.

2. UNFALSIFIED CONTROL APPROACH

The discrete-time feedback system of interest is as follows:

\[ y(t) = \Pi[\delta u(t)](t), \quad t \in \mathbb{Z}_+ := \{0, 1, \ldots\} \]

\[ \delta u(t) = K_{\sigma(t)}[r - y](t) \]

where \( \Pi \) denotes an uncertain strictly causal LTI plant with input increment \( \delta u \) and output \( y \), and \( r \) denotes the reference to be tracked by the plant output \( y \).\(^{\dagger}\) Here, \( K_{\sigma(t)}[y - r](t) \) has to be intended as a shorthand notation for \( K_{\sigma(t)}[y - r](t) \). The latter means that all the \( N \) causal candidate controllers are fed at all times by the tracking error \( e := r - y \), and the output of the \( \sigma(t) \) controller is used as the input-increment to the plant at time \( t \). Though unnecessary for stability analysis \(^{17}\), hereafter it is assumed that \( \Pi \) belongs to a parametric uncertainty set \( \mathcal{P} \), consisting of a family of finite-dimensional LTI plants \( \Pi = \Pi(\theta) \) with transfer functions \( \Pi(d, \theta) = B(d, \theta)/A(d, \theta) \)

\[ \mathcal{P} := \{\Pi(\theta), \theta \in \Theta\} \]

where \( A(d, \theta) \) and \( B(d, \theta) \) denote polynomials with strictly Hurwitz greatest common divisor (g.c.d.), and \( A(d, \theta) \) monic, viz. \( A(0, \theta) = 1 \). Plant input increments are considered throughout the paper so as to address the common practice of controllers equipped with an 'integral action', which implies that \((1 - d)\) divides \( A(d, \theta) \), viz. \( A(d, \theta) = a(d, \theta)(1 - d) \), \( \theta \in \Theta \), for some monic polynomial \( a(d, \theta) \). Finally, strict causality of \( \Pi(\theta) \) amounts to \( B(0, \theta) = 0 \). For the sake of brevity, from now on, the parameter vector \( \theta \) will be omitted in all polynomials, unless otherwise required for avoiding possible confusion.

The device responsible for orchestrating \( \sigma(t) \) is the so-called switching supervisor, and (1), combined with such a device, is referred to as an ASC system denoted \( \Pi(\hat{K}_{\sigma(\cdot)}) \).\(^{\dagger}\) Let \( \mathcal{S} \) denote the linear space of all the real-valued sequences on \( \mathbb{Z}_+ \). Given a vector-valued sequence \( x \in \mathcal{S} \) of dimension \( n \), \( x^t \) denotes its time truncation up to time \( t \), i.e. \( x^t := \{x(0), x(1), \ldots, x(t)\} \), with

\(^{\dagger}\)Here, \( H[\cdot] \) denotes a causal linear operator with transfer function \( H(d) = M(d)/N(d) \), with \( M \) and \( N \) polynomials in the unit backward shift operator \( d \). The streamlined notation \( M(d)v(t) = N(d)u(t), \quad M(d) = \sum_{k=0}^\infty n_k d^k, \quad N(d) = \sum_{k=0}^\infty n_k d^k \), will be adopted to indicate the difference equation \( m_0 v(t) + m_1 v(t-1) + \cdots + m_p v(t-p) + \cdots + n_0 u(t-\hat{N}) \).

\(^{\dagger}\)There are many ways for implementing the adaptive controller \( K_{\sigma(t)} \). The state of \( K_{\sigma(t)} \) at the time \( t \) at which it is switched into the loop can be either arbitrarily initialized or, perhaps, initialized by bumpless control transfer techniques \(^{19}\), so as to reduce the switching transients. Controller implementation, though important \(^{20}\), is not crucial for addressing synthesis issues of test functionals. Thus, for the sake of simplicity, a common state multicontroller scheme \(^{21}\) will be adopted: Consider as a shared state the vector

\[ z(t) := [\xi(t - 1), \xi(t - p), \delta u(t - 1), \ldots, \delta u(t - p)]^T \]

where \( p \) denotes the maximum order among all the controllers in \( \kappa \). Accordingly, the output of \( K_{\sigma(t)} = K_i \) at time \( t \) is given by

\[ \delta u(t) = K_i[z](t) := [s_{1i} \ldots s_{pi} - r_{1i} \ldots - r_{pi}]^T z(t) + s_{0i} \xi(t) \]

where \( s_{ij} \) and \( r_{jk} \), \( i \in \mathbb{N} \), \( j + 1 = \hat{p} + 1 \) and \( k \in \mathbb{P} \), are the coefficients of \( S_i(d) = \sum_{n=0}^p s_{ni} d^n \) and, respectively, of \( R_i(d) = 1 + \sum_{r=1}^{p-r} f_{rn} d^n \), being \( K_i(d) := S_i(d)/R_i(d) \). In this way, \( K_{\sigma(t)} \) reduces to a single controller with adjustable parameters, with no need of implementing all the candidate controllers in \( \kappa \).

Lemma 1

\[ x(k) \in \mathbb{R}^n. \] Let \( \ell_p(\mathbb{Z}_+) \), \( p \geq 1 \), be the space of all vector-valued real sequences with bounded \( \ell_p \)-norm defined as:

\[
\| x \|_p := \sum_{k=0}^{\infty} |x(k)|^p = \sum_{k=0}^{\infty} \sum_{i=1}^{n} |x_i(k)|^p, \quad \| x \|_\infty := \sup_k |x(k)|
\]

(E.g., \( \ell_2(\mathbb{Z}_+) \) and \( \ell_\infty(\mathbb{Z}_+) \) denote the linear spaces of sequences belonging to \( \mathbb{S} \) with bounded energy, respectively, peak. Another important class of signals is finite-power signals. In such a case, we denote with \( P(\mathbb{Z}_+) \) the linear space consisting of those sequences \( x \in \mathbb{S} \) with bounded \( P \)-norm defined as:

\[
\| x \|_P := \sup_{t \in \mathbb{Z}_+} \frac{1}{t+1} \sum_{k=0}^{t} |x(k)|^2 = \sup_{t \in \mathbb{Z}_+} \frac{1}{t+1} \sum_{k=0}^{t} \sum_{i=1}^{n} |x_i(k)|^2
\]

Hereafter, \( \| \cdot \| \) will denote a generic signal norm among the ones defined in (5) and (6), and \( N(\mathbb{Z}_+) \) its related normed space. The following notion of input–output stability is adopted:

**Definition 1**

A causal LTI system \( H \) with \( m \) inputs \( w_i \) and \( p \) outputs \( z_j \) is said input–output \( N \)-stable if for any \( w_i \in \mathbb{S}, i \in \mathbb{N} \), there exist finite nonnegative constants \( \alpha_j, \beta_j \) such that

\[
\| z_j \| \leq \sum_{i \in \mathbb{N}} \alpha_j \| w_i \| + \beta_j \quad \forall t \in \mathbb{Z}_+ \quad \forall j \in \mathbb{P}
\]

where \( z_j \) denotes the system output response to the inputs \( w_i \).

The constants \( \beta_j \) allow for consideration of non-zero initial states. It should be emphasized that the stability of system \( H \) requires that (7) holds true, possibly with different constants \( \alpha_j, \beta_j \), for any input. From now on, the ‘output’ data

\[
z(t) := (\Pi/K_{\sigma(t)})(r(t) = [\delta u(t)y(t)])
\]

will be intended as the I/O pairs of the uncertain plant \( \Pi \) in (1), while the ‘input’ data \( w \) will represent all exogenous inputs. Given the set \( K \) of \( N \) candidate controllers, the main task of the control system is to ensure that \( (\Pi/K_{\sigma(t)}) \) be \( N \)-stable, \( \forall \Pi \in \mathbb{P} \). In order that such a problem be well-posed, the following minimal requirement is assumed.

* Problem Feasibility: For every \( \Pi \in \mathbb{P} \), there are indices \( i, i \in \mathbb{N} \), such that \( (\Pi/K_i) \) is internally stable.

In order to quantify the suitability of the candidate controller \( K_i \) in the potential feedback loop \( (\Pi/K_i) \), \( N \) nonnegative test functionals \( V_i(t) = \lambda_i(t, z') \), \( i \in \mathbb{N} \), are used. Accordingly, given \( z' \), the supervisor compares, at every \( t \in \mathbb{Z}_+ \), the values taken on by the \( N \) test functionals and selects the controller index \( \sigma(t) \) via the following hysteresis switching logic (HSL):

\[
\sigma(t+1) = \arg \min_{i \in \mathbb{N}} \{ V_i(t) - \delta_i,\sigma(t) \}, \quad \sigma(0) = \sigma_0
\]

where \( \sigma_0 \) is arbitrary in \( \mathbb{N} \), \( \delta_i, \sigma(t) \) is Kronecker’s index, and \( h > 0 \) denotes a hysteresis constant. In MMUASC and UASC system analysis, an important role is played by the next lemma. Let \( \mathbb{E} \) denote the class of all possible switching functions \( s: \mathbb{Z}_+ \rightarrow \mathbb{N} \) giving rise to the switched system \( (\Pi/K_{\sigma(t)}) \). Consider the assumptions:

**A1.** For each \( s \in \mathbb{E} \) and \( i \in \mathbb{N} \), \( V_i(t) \) admits a limit (even infinite) as \( t \rightarrow \infty \);

**A2.** There is at least one integer \( m \in \mathbb{N} \) such that \( V_m(\cdot) \) is bounded for each \( s \in \mathbb{E} \).

**Lemma 1 (HSL-Lemma, Morse et al. [22])**

Consider the ASC system (1) with \( \sigma(t) \) selected in accordance with the HSL (8). Then, if A1 and A2 hold, for any initial condition and any reference \( r \), there is a finite time \( t_f \) such that \( \sigma(t) = f \), \( f \in \mathbb{N} \), \( t \geq t_f \). Moreover, \( V_f(\cdot) \) is bounded. □
2.1. Need for suitable data filtering

Test functionals, satisfying A1 and A2 and whose boundedness yields input–output $\mathcal{N}$-stability to the adaptive system (1), have been investigated [1, 17, 18, 23]. A convenient way for constructing $V_i$’s satisfying the stated assumptions is as follows: First, select a nonnegative performance/identification functional $J_i(t)=\mathcal{J}(t, z^i), i \in \overline{N}$, apt to quantify the suitability of the controller $K_i$ relatively to the plant $\Pi$ to be controlled. In order to be able to deduce $\mathcal{N}$-stability of $(\Pi/K_i)$ from boundedness of $J_i(\cdot)$, $J_i$ has to satisfy the boundedness condition A2 (with $V_m$ replaced by $J_0$) provided that $(\Pi/K_i)$ be $\mathcal{N}$-stable; Second, the test functional $V_i$ to be used in the switching logic (8) is set so as to ensure A1, e.g.

\[
V_i(t) := \max\{J_i(\tau), \tau = t_0, \ldots, t\}, \quad t_0 \geq 0
\]  

(9)

**Definition 2**

Given a causal LTI operator with transfer function $H(d) = F(d)/G(d)$ with input $\xi$ and output $\eta$, the notation $\tilde{\eta} = \mathcal{H}_0[\tilde{\xi}]$ means that $\eta(t)$ is the solution over $\mathbb{Z}_+$ of the following difference equation:

\[
G(d)\tilde{\eta}(t) = F(d)\tilde{\xi}(t), \quad \eta(k) = \tilde{\xi}(k) = 0, \quad k = -1, -2, \ldots
\]  

\[
\Box
\]

In UASC and MMUASC systems, performance/identification functionals hinge upon the so-called virtual-reference (VR) tool. For each $i \in \overline{N}$, the variable

\[
v_i(t) = y(t) + K_i^{-1}[\delta u](t), \quad t \in \mathbb{Z}_+
\]  

(10)

is computed in real-time, provided that $K_i$ be causal, and stably causally invertible (CSCI)\(^\dagger\). In words, $v_i$ equals the fictitious or virtual reference that, if injected into the feedback system $(\Pi/K_i)$, reproduces $z(\cdot)$ under the mentioned initial state condition. Hence, if $(\Pi/K_{\sigma(i)})$ represents the linear (time-varying) transformation (1) mapping $r$ into $z$, one has

\[
z(t) = (\Pi/K_{\sigma(i)})[r](t) = (\Pi/K_i)[v_i](t)
\]  

(11)

where $(\Pi/K_i)$ denotes the LTI transformation consisting of the plant $\Pi$ fed-back by the $i$th candidate controller $K_i$. The first use of the VR in ASC dates back to the UASC of [16, 17], where $J_i$ in (9) was set equal to $J_i^S$ as in (12) so as to provide an estimate from below of the induced gain of the $\ell_2 - \ell_2$ map embodied by $(\Pi/K_i)$ from $v_i$ to $[\delta u - v_i]'$, viz.

\[
J_i^S(t) := \frac{\|([\delta u - v_i]'')^2}{\|v_i''\|^2} = \frac{\|([\Sigma_{s/i}]_0[v_i]'')^2}{\|v_i''\|^2}, \quad i \in \overline{N}
\]  

(12)

where the last equality holds under the zero initial conditions constraint. In (12),

\[
\Sigma_{s/i}(d) := \frac{1}{1+\Pi(d)K_i(d)} \begin{bmatrix} K_i(d) \\ 1 \end{bmatrix}
\]  

(13)

denotes the mixed sensitivity of $(\Pi/K_i)$ UASC systems equipped with simple test functionals such as (12), wherein no explicit use is made of a priori dynamic models approximating $P$, are significant devices, particularly from a conceptual viewpoint, in that they ensure $\mathcal{N}$-stability of $(\Pi/K_{\sigma(i)})$ under the only requirement of Problem Feasibility [17, 23].

However, there are further aspects in an ASC system which deserve consideration. Particularly, an important quality factor of a test functional is its capability of inferring if a candidate controller would make the loop unstable once switched-on in feedback to the plant. Given any $\Pi \in \mathcal{P}$, let $\mathcal{S}(\Pi) \subseteq \overline{N}$ be the set of all indices $s \in \overline{N}$ such that $(\Pi/K_s)$ is stable, and $S^c(\Pi)$ the complement

\(^\dagger\)The case of non-CSCI controllers is not discussed in this paper, because, as shown in Section 4, the most effective multi-model-based ASC schemes, though conceptually hinging upon the VR entity, are implementable without an explicit VR-computation, and, hence, apply to the generic controller case.
of $S(\Pi)$ in $\overline{N}$. By assuming that a stabilizing controller $K_s$ is interconnected in feedback to $\Pi$ at time $t$, a necessary condition for avoiding that $K_s$ be substituted at time $t+1$ by a destabilizing controller is

$$J_i(t)<J_i(t)+h, \quad i \in S^c(\Pi)$$

(14)

Unfortunately, (14) cannot be ensured by functionals deriving from (12) whereby no plant model is used. In fact, $K_i[u_i-y_i(t)]=\delta u(t), \forall i \in \overline{N}$. Hence, under the assumption that $K_i$ be CSCI, (12) turns out to be bounded if $\delta u$ is such, irrespective of stability of $(\Pi/K_i)$, that (14) need not hold as destabilizing controllers out of the loop yield bounded test functionals.

3. REFERENCE-LOOP IDENTIFICATION AND MMUASC SYSTEMS

As remarked in Section 2, an effective inference of the suitability of a potential $K_i$ as a feedback controller for an uncertain plant $\Pi$ cannot be achieved if no information on $\Pi$ is extracted from $z$. Hereafter, it will be assumed that a finite family $\mathcal{M}:=\{M_i, i \in \overline{N}\}$ of $N$ strictly causal LTI dynamic models $M_i$ is available,

$$M_i(d):=B_i(d)/A_i(d), \quad i \in \overline{N}$$

where $B_i(d)$ and $A_i(d)$ are polynomials with strictly Schur g.c.d., $A_i(d)=a_i(d)(1-d)$ and $a_i(0)=1$. In an MMUASC scheme, the candidate controllers $K_i, i \in \overline{N}$, are chosen so as to satisfy at least the feasibility condition, while the $M_i$’s, along with the associated $K_i$’s, form a finite family $\mathcal{R}:=\{(M_i/K_i), \ i \in \overline{N}\}$ of internally stable feedback-loops, each designed to fulfill desirable prescriptions. Such prescriptions might only be of qualitative type, and, hence, $K_i$, in general, need not be optimal with respect to any specific performance index. Hereafter, $(M_i/K_i)$ will be referred to as the $i$th ‘tuned-loop’ or ‘reference-loop’. Given an unknown plant $\Pi \in \mathcal{P}$, one of the main steps in MMUASC is to carry out a reference-loop identification task, viz., select a candidate controller $K_\sigma$ in such a way that $(\Pi/K_\sigma)$ behave as closest as possible to one of the candidate reference-loops in $\mathcal{R}$. Hence, roughly speaking, the ideal goal of the switching supervisor, can be envisaged as follows. Given an uncertain plant $\Pi \in \mathcal{P}$, find an index $\sigma \in \overline{N}$ such that:

(i) $(\Pi/K_\sigma)$ be stable;

(ii) The behavioral data produced by $(\Pi/K_\sigma)$ in response to $r$ be as close as possible to the ones produced by $(M_\sigma/K_\sigma)$ in accordance to the reference-loop identification criterion**

$$\sigma := \arg \min_{\overline{N}} \sup_{r \neq 0} \frac{||(\Pi/K_i)[r]-(M_i/K_i)[r]||}{\|F_i[r]\|}$$

(15)

where, by the sake of simplicity no time-argument is shown, and $F_i$ denotes an LTI filter such that $r \neq 0$ implies $\|F_i[r]\|>0$.

On-line implementation of (15) is impossible without using either pre-routing or $N$ identical copies of the unknown plant. The latter instance is unrealistic, while the former, in general, has to be ruled out because it typically causes large and long-lasting learning transients. A way for side-stepping such a difficulty hinges upon the use of the VR’s (10) in place of $r$. Let $(M_i/K_i)$ be driven by $u_i$. Then, the variables $y_{i|i}$ and $\delta u_{i|i}$ are as follows:

$$y_{i|i}(t)=M_i[\delta u_{i|i}]\hat{}(t), \quad t \in \mathbb{Z}_+$$

$$\delta u_{i|i}(t)=K_i[u_i-y_{i|i}]\hat{}(t)$$

(16)

**As in (12), the rationale for considering identification criteria expressed in terms of ratios of norms stems from the fact that functionals related to (15) make it possible to assess the suitability of a candidate controller $K_i$, whenever relevant behavioral features of $(\Pi/K_i)$ depend on feedback system induced gains.
Accordingly, by letting \((M_i/K_i)[v_i] := [\delta u_{i/i} y_{i/i}]'\), (15) is modified in the following on-line implementable form:

\[
\sigma := \arg \min_{i \in \mathcal{N}} \sup_{v_i \neq 0} \frac{\|\Pi/K_i[v_i] - (M_i/K_i)[v_i]\|}{\|F_i[v_i]\|}
\]

(17)

**Remark 1**

In the sequel, in order to simplify notations, unless otherwise needed to avoid possible ambiguities, initial conditions for all the operators in (17), though not explicitly indicated, are understood to be zero. The general case of arbitrary initial conditions will be addressed in Section 4.

Therefore, by letting \(z_{i/i} := (M_i/K_i)[v_i] = [\delta u_{i/i} y_{i/i}]'\) (Figure 2) and \(\tilde{z}_i(t) := z(t) - z_{i/i}(t), \quad i \in \mathcal{N}\) test functionals related to the identification criterion (17) are as follows:

\[
J_i(t) := \frac{\|\tilde{z}_i(t)\|}{\|(F_i[v_i])'\|}, \quad i \in \mathcal{N}
\]

(18)

The \(i\)th reference-loop identification functional \(J_i(t)\) in the form (18) will be also referred to as the \(F_i\)-normalized discrepancy at time \(t\) between \((\Pi/K_i)\) and \((M_i/K_i)\). As will be seen soon, the choice of \(F_i\) plays a crucial role. In fact, on-line computation of (18) (Figure 3) is in practice the reason being the one elaborated in Remark 3.

Natural functionals for reference-loop identification (Figure 3) are the ones characterized by the following \(F_i\)-choices.

- \(F_i = I\) yields a test functional \(V_i\) which is an estimate from below of the induced gain of \((\Pi/K_i) - (M_i/K_i)\) mapping \(v_i\) into \(\tilde{z}_i\). Such a choice turns out not to be an effective one, the reason being the one elaborated in Remark 3.

- \(F_i = (\Pi/K_i) - [0I]'\), viz.

\[
F_i[v_i](t) = [\delta u(t)y(t) - v_i(t)]' =: \eta_{a/i}(t)
\]

yields what will be referred to as the PA-percentage discrepancy relatively to the potential loop performance data \(\eta_{a/i}\). As will be shown soon, its related test functional \(V_i^{PA}\) is an estimate from below of the induced gain of the operator (19). The reason for the adopted terminology stems from the fact that the normalization factor involves the potential (P) loop \((\Pi/K_i)\) and the operator (19) depends on the additive (A) uncertainty on \(\Pi\) relatively to \(M_i\), the latter considered as the plant nominal model;

- \(F_i = (\Pi/K_i)\), viz. \(F_i[v_i](t) = z(t)\) yields what will be referred to as the PC-percentage discrepancy relatively to the potential loop data \(z\). As will be seen, its related test functional \(V_i^{PC}\) is an estimate from below of the induced gain of the operator (25). Here, the normalization factor involves the potential (P) loop \((\Pi/K_i)\), while the operator (25) depends on the coprime (C) factor uncertainty on \(\Pi\) relatively to \(M_i\);
• $F_i = (M_i/K_i) - [0I]^T$, viz.

$$F_i[v_i](t) = [\delta u_{i/j}(t) y_{i/j}(t) - v_i(t)]^T =: \eta_{i/j}(t)$$

yields what will be referred to as the RA-percentage discrepancy relatively to the reference-loop data $\eta_{i/j}$. As will be seen, its related test functional $V_i^{RA}$ is an estimate from below of the induced gain of the operator (29). In such a case, the normalization factor involves the reference (R) loop ($M_i/K_i$), while the operator (29) depends on the additive (A) uncertainty on $\Pi$ relatively to $M_i$.

• $F_i = (M_i/K_i)$, viz. $F_i[v_i](t) = z_{i/j}(t)$ yields what will be referred to as the PC-percentage discrepancy relatively to the potential loop data $z_{i/j}$. As will be seen, its related test functional $V_i^{RC}$ is an estimate from below of the induced gain of the operator (33). In such a case, the normalization factor involves the reference (R) loop ($M_i/K_i$), while the operator (33) depends on the coprime (C) factor uncertainty on $\Pi$ relatively to $M_i$.

In the sequel, both functionals $J_i^{PA}$ and $J_i^{PC}$ whose denominators involve the $i$th potential loop, will be referred to as P-percentage discrepancies, while $J_i^{RA}$ and $J_i^{RC}$, with normalizations involving the $i$th reference-loop will be denoted as R-percentage discrepancies.

### 3.1. P-percentage discrepancies

In accordance with (11), given the reference-loop ($M_i/K_i$) (16), the potential loop ($\Pi/K_i$) can be conceived as a perturbation of ($M_i/K_i$), the perturbation being caused by the possible difference between $\Pi$ and $M_i$. By P-percentage discrepancies, it is possible to test necessary conditions for instability of a potential loop ($\Pi/K_i$) based on small-gain stability arguments: the specific $F_i$-choice in (18) (yielding either $z$ or $\eta_{a/j}$) depends on the adopted description of the plant uncertainty.

Let the uncertain plant $\Pi$ be represented in terms an additive perturbation $\Pi - M_i$ of the nominal model $M_i$. A well-known result of small-gain stability [24] is that a sufficient condition for $\mathcal{N}$-stability of ($\Pi/K_i$) is that $\|\Xi_{a/i}\|_{\text{ind}} < 1$, where

$$\Xi_{a/i}(d) := \frac{(\Pi(d) - M_i(d))K_i(d)}{1 + M_i(d)K_i(d)} = \frac{(B(d)A_i(d) - B_i(d)A(d))S_i(d)}{A(d)\zeta_{i/j}(d)}$$

with $\zeta_{i/j} := A_iR_i + B_iS_i$, the characteristic polynomial of ($M_i/K_i$). It can be seen that, under the zero initial condition constraint, one has $\bar{z}_i(t) = (\Xi_{a/i})_0[\eta_{a/i}](t)$. Hence, in order to get an estimate from below of $\|\Xi_{a/i}\|_{\text{ind}}$, one can set $F_i[v_i] := \eta_{a/i}$, yielding

$$J_i^{PA}(t) := \frac{\|\bar{z}_i(t)\|}{\|\eta_{a/i}\|} = \frac{\|((\Xi_{a/i})_0[\eta_{a/i}])^f\|}{\|\eta_{a/i}\|}, \quad i \in \mathcal{N}$$

Figure 3. Block diagram of MMUASC for reference-loop identification.
In view of (19), in order to be able to deduce \( \mathcal{N} \)-stability of \((\Pi/K_i)\) from the magnitude of \( J_i^{\text{PA}} \), the polynomial \( A_i(d) \) must cancel all the (possible) unstable roots of \( A(d) \). Consequently, the following assumption is needed.

**B1.** The denominator \( A(d, \theta) \) of \( \Pi \in \mathcal{P} \) satisfies

\[
A(d, \theta) = A^+(d)A^-(d, \theta)
\]

with \( A^+(d) \) fixed and known, and \( A^-(d, \theta) \) strictly Hurwitz.

Under B1, it is possible to select each nominal model \( M_i \) in such a way that \( A_i(d) := A^+(d)A_i^-(d) \), with \( A_i^-(d) \) strictly Hurwitz. Hence, \( M_i(d) = M_i^-(d)/A^+(d) \), and

\[
\Pi(d) = \frac{1}{A^+(d)}(M_i^-(d) + \Delta_i(d)), \quad M_i^-(d) := B_i(d)/A_i^-(d)
\]

Therefore, if each \( A_i(d) \) is selected in such a manner, \( \Xi_{s/i}(d) \) turns out to be

\[
\Xi_{s/i}(d) := A_i^-(d)S_i(d)\tilde{Y}_{s/i}^{-1}(d)\Delta_i(d) := H_i(d)\Delta_i(d)
\]

Thus, under B1, \( J_i^{\text{PA}} \) can be used as an estimate from below \( \|\Xi_{s/i}\|_{\text{ind}} \). Since \( \|\Xi_{s/i}\|_{\text{ind}} \geq 1 \) is a necessary condition for instability of \((\Pi/K_i)\), \( \mathcal{M} \) must be set dense enough in \( \mathcal{P} \), so as to ensure that, for any \( \Pi \in \mathcal{P} \), there exist indices \( i \in \mathcal{N} \), yielding stable loops \((\Pi/K_i)\) such that \( \|\Xi_{s/i}\|_{\text{ind}} \leq \beta < 1 \).

**Proposition 1**

Let \( \Theta \) be a compact set, and \( \theta \mapsto \Pi(\theta) \) continuous on \( \Theta \). Then, under condition B1, for any positive real \( \beta \), there always exists a finite model distribution \( \mathcal{M} \) such that

\[
\max_{\Pi \in \mathcal{P}} \min_{i \in \mathcal{N}(\mathcal{P})} \|\Xi_{s/i}\|_{\text{ind}} =: \beta_{\text{PA}} \leq \beta
\]

**Proof**

Notice that the induced norm on \( \mathcal{N}(\mathbb{Z}_+) \) of a finite-dimensional LTI stable operator (i.e. with absolutely summable pulse response) can be upper-bounded by its \( \mathcal{H}_\infty \) norm (see [24]), which depends continuously on \( \theta \). Thus, under continuity assumption of the map \( \theta \mapsto \Pi(\theta) \), for every \( \theta \in \Theta \) and \( \beta \in \mathbb{R}_+ \), there exists an open ball \( \mathcal{B}_\beta(\theta) \) around \( \theta \) such that \( \|\Xi_{s/i}\|_{\text{ind}} \leq \beta \) (e.g. see [25]). The existence of a finite covering follows from compactness of \( \Theta \).

A model distribution, for which such a property holds, will be referred to as a \( \beta_{\text{PA}} \)-dense model distribution and denoted by \( \mathcal{M}(\beta_{\text{PA}}) \). Further, assuming \( \mathcal{M} = \mathcal{M}(\beta_{\text{PA}}) \) and \( \Pi \in \mathcal{P} \) given, a specific potential loop \((\Pi/K_i)\) is said to have the \( \beta_{\text{PA}} \)-property if \( \|\Xi_{s/i}\|_{\text{ind}} \leq \beta_{\text{PA}} \).

An analysis similar to the one considered for the additive uncertainty (21) can be carried out in case \( \Pi \) be represented in terms of a coprime factor uncertainty

\[
(A_i(d) - \tilde{A}_i(d)) \quad y(t) = (B_i(d) + \tilde{B}_i(d))\delta u(t)
\]

which yields

\[
A_i(d)y(t) = B_i(d)\delta u(t) + e_i(t)
\]

where \( e_i \) is the equation error given by \( e_i(t) = \tilde{A}_i(d)y(t) + \tilde{B}_i(d)\delta u(t) \) (Figure 4).

![Figure 4. Coprime factors model error representation.](image-url)
Let now \( K_i \) be a controller which makes \((M_i / K_i)\) internally stable. As beforehand, one uses a result of small-gain stability. Given an uncertainty \( \tilde{L}_i(d) := [\tilde{B}_i(d) \, \tilde{A}_i(d)] \) on the plant coprime factors, a sufficient condition for \( \mathcal{N}\)-stability of \((\Pi/K_i)\) is that \( \|\Psi_{s/i}\|_{\text{ind}} < 1 \), where

\[
\Psi_{s/i}(d) := \frac{[\sum S_i(d) \, R_i(d)]}{\chi_{i-i}(d)} [\tilde{B}_i(d) \, \tilde{A}_i(d)] := Q_{i-i}(d)\tilde{L}_i(d)
\]  

(25)

Now, it can be seen that \( \tilde{z}_i := (\Psi_{s/i})_0[z] \). Thus, the problem of inferring stability of a candidate loop \((\Pi/K_i)\) can be approached by setting in (18) \( F_i[v_i] := z \), which leads to

\[
f_i^{PC}(t) := \frac{\|z'_i(t)\|}{\|z'_i(t)\|} = \frac{\|((\Psi_{s/i})_0[z])'\|}{\|z'_i(t)\|}, \quad i \in \mathbb{N}
\]  

(26)

Since \( \|\Psi_{s/i}\|_{\text{ind}} \geq 1 \) is a necessary condition for instability of \((\Pi/K_i)\), \( \mathcal{M} \) must be set dense enough in \( \mathcal{P} \), so as to ensure that, for any \( \Pi \in \mathcal{P} \), there are indices \( i \in \mathbb{N} \) yielding stable loops \((\Pi/K_i)\) with \( \|\Psi_{s/i}\|_{\text{ind}} \leq \beta < 1 \). In accordance with the next proposition, (26) has the advantage over (20) of handling also unknown plant unstable perturbations.

**Proposition 2**

Let \( \Theta \) be a compact set, and \( \theta \to \Pi(\theta) \) continuous on \( \Theta \). Then, for any positive real \( \beta \), there always exists a finite model distribution \( \mathcal{M} \) such that

\[
\max_{\Pi \in \mathcal{P}} \min_{i \in \mathbb{N}} \|\Psi_{s/i}\|_{\text{ind}} =: \beta_{PC} \leq \beta
\]  

(27)

According to Propositions 1 and 2, boundedness of (20) and (26) holds irrespective of the stability of \((\Pi/K_i)\). Thus, in order to ensure, via either (20) or (26), falsification of destabilizing controllers, one needs to set the model distribution dense enough in \( \mathcal{P} \), viz. \( \beta_{PA} \) (or \( \beta_{PC} \)) sufficiently small. To this end, consider the ASC system in (1) under the HSL (8), with

\[
V_i(t) = \begin{cases} 
0 & \text{if } \|z'_i(t)\| = 0 \\
\max\{J_i(\tau), \tau = 0, \ldots, t\} & \text{otherwise}
\end{cases}
\]  

(28)

Next theorem captures the basic properties ensured by the P-percentage functionals.

**Theorem 1**

Consider the ASC system \((\Pi/K_{\sigma(t)})\) (1), \( \Pi \in \mathcal{P}, \mathcal{P} \) compact, under zero initial conditions. Let \( \sigma(t) \) be selected in accordance with the HSL (8), with \( V_i(t) \) as in (28). Then, for any reference \( r \in \mathbb{S} \), the following properties hold:

(i) Let \( J_i(t) := J_i^{PA}(t)[J_i^{PC}(t)] \). Under a model distribution \( \mathcal{M}(\beta_{PA}), \beta_{PA} < 1 - h, \beta_{PA} \) as in (23) \([\mathcal{M}(\beta_{PC}), \beta_{PC} < 1 - h, \beta_{PC} \) as in (27)], the switching stops in finite time and \((\Pi/K_{\sigma(t)})\) are \( \mathcal{N}\)-stable;

(ii) Under the same conditions as in (i), the total number of switches \( N_{\sigma} \) is bounded by

\[
N_{\sigma} \leq \frac{N}{[\beta_{PA}/h][N_{\sigma}]} [N_{\sigma} \leq \frac{N}{[\beta_{PC}/h]}], \quad [x] \text{ denotes the smallest integer greater than or equal to } x \in \mathbb{R}_+; \text{ the occurrence of the condition } J_i^{PA}(t) \geq \beta_{PA}[J_i^{PC}(t) \geq \beta_{PC}], t \geq t_0, \forall i \in \mathbb{S}'(\mathcal{P}),
\]

ensures that no destabilizing controllers will be switched-on at and after time \( t \); Finally, if \( h \geq \beta_{PA}/[\beta_{PC}] \), each candidate controller can be switched-on no more than once, and any stabilizing controller with the \( \beta_{PA}[\beta_{PC}] \)-property, once switched-on in feedback with the plant, will be never switched-off thereafter.

**Proof**

See the Appendix.

**Remark 2**

\( \mathcal{N}\)-stability of \((\Pi/K_{\sigma(t)})\) does not imply that the same property hold for the final loop \((\Pi/K_f)\). In fact, Theorem 1 does not state that the switching index \( \sigma(t) \) stops on a \( \beta \)-property controller,
nor to an $\mathcal{N}$-stabilizing controller. It only states that $\mathcal{N}$-stability of $(\Pi/K_f)$ is unfalsified by the experimental data. Nonetheless, under a $\beta$-dense model distribution, the maximum value taken on by the test functional of the final controller is less than (within the tolerance $h$) the ones related to the $\beta$-property controllers.

3.2. R-percentage discrepancies

A natural and effective way to infer controller suitability from data $z$ consists of endowing a test functional $V_i$ having the property that $(\Pi/K_i)$ is $\mathcal{N}$-stable if and only if $V_i$ takes on finite values, $\forall r \in \mathcal{N}(\mathbb{Z}_+)$ and $\sigma \in \mathcal{S}$. Section 3.1 showed that this is not the case for P-percentage discrepancies. As shown next, the above desired property for $V_i$ holds for R-percentage discrepancies. The specific $F_i$-choice in (18) (here either $z_{i|i}$ or $\eta_{i|i}$) depends on the adopted description of the plant uncertainty.

Let the uncertain plant $\Pi$ be represented in terms of the nominal model $M_i$ and its related additive uncertainty. In Section 3.1 it was shown that, under condition B1 (see comments before (19)), it is possible to select $\mathcal{M}$ in such a way that, by letting $F_i[v_i] := \eta_{i|i}$, the functional $J_{PA}^i$ provides a margin for stability of the potential loop $(\Pi/K_i)$. Now, under B1 and the same choice for the $M_i$’s as in (21), for the map from $\eta_{i|i}$ to $z_i$ one has $z_i(t) = -(\Lambda_{x/i})_0[\eta_{i|i}](t)$, where

$$\Lambda_{x/i}(d) := A^-(d) S_i(d) \chi^{-1}_{x/i}(d) \Delta_i(d) := H_{x/i}(d) \Delta_i(d)$$

with $\chi_{x/i} := AR_i + BS_i$, the characteristic polynomial of $(\Pi/K_i)$. Accordingly, by letting $F_i[v_i] := \eta_{i|i}$, one has

$$J_{RA}^i(t) := \frac{\|z_i^f\|}{\|\eta_{i|i}\|} = \frac{\|((\Lambda_{x/i})_0[\eta_{i|i}]^f\|}{\|\eta_{i|i}\|}, \quad i \in \bar{N}$$

Note that, in contrast with $J_{PA}^i$, boundedness of $J_{RA}^i$ depends on stability of $(\Pi/K_i)$, viz. a potential candidate loop $(\Pi/K_i)$ is stable if and only if $J_{RA}^i$ takes on finite values, $\forall r \in \mathcal{N}(\mathbb{Z}_+)$ and $\sigma \in \mathcal{S}$.

Proposition 3
Let $\Theta$ be a compact set, and $\theta \rightarrow \Pi(\theta)$ continuous on $\Theta$. Then, under condition B1, for any positive real $\beta$, there always exists a finite model family $\mathcal{M}$ such that

$$\max_{\Pi \in \mathcal{P}} \min_{i \in \mathcal{S}(\mathcal{P})} \|\Lambda_{x/i}\|_{\text{ind}} =: \beta_{RA} \leq \beta$$

Remark 3
A conceivable alternative to $J_{RA}^i(t)$ consists of selecting the identification functional so as to provide an estimate from below of the induced gain of the map from $v_i$ to $z_i$. However, as the reader may check, this yields

$$J_i(t) := \frac{\|z_i^f\|}{\|v_i^f\|} = \frac{\|((\Lambda_{x/i}[\Sigma_{i/i}]_0[v_i])^f\|}{\|v_i^f\|}$$

where $\Sigma_{i/i}$ is the mixed sensitivity matrix of $(M_i/K_i)$, viz. (13) with $\Pi$ changed into $M_i$. Such a choice need not be an effective one, since the upper-bound on $J_i$ turns out to be dependent (not normalized) on $\|\Sigma_{i/i}\|_{\text{ind}}$. In turns, such a magnitude might be quite different at the various $i$’s, and, hence, introduce undesirable biases in the switching mechanism.

A convenient way for dealing with possibly unknown unstable modes of the plant, and getting rid of B1 is to replace the denominator of $(30)$ with $F_i[v_i] := z_{i|i}$. In fact, let $\Pi$ be represented in terms of a coprime factor uncertainty. Following similar lines as in Section 3.1, one can verify that, under the zero initial condition constraint, the transfer matrix from $z_{i|i}$ to $\tilde{z}_i$ coincides with

$$\Omega_{x/i}(d) := \left[ -S_i(d) \begin{bmatrix} R_i(d) & \tilde{B}_i(d) \tilde{A}_i(d) \end{bmatrix} \right] := Q_{x/i} \tilde{L}_i$$
where \( \bar{L}_i = \hat{B}_i \hat{A}_i \) denotes a stable uncertainty on plant model coprime factors. Thus, \( F_i[v_i] := z_{i/i} \) yields

\[
J_i^{\text{RC}}(t) = \frac{\|z_{i/i}^f\|}{\|z_{i/i}^r\|} = \frac{\|((\Omega_{s/i})_{0} z_{i/i})^f\|}{\|z_{i/i}^r\|}, \quad i \in \overline{N}
\] (34)

Similar to \( J_i^{\text{RA}} \) but in contrast with \( J_i^{\text{PC}} \), boundedness of \( J_i^{\text{RC}} \) depends on the stability of \((\Pi/K_i)\). As a result of next proposition, any finite \( \beta_{\text{RC}} \) (instead of \( \beta_{\text{PC}} < 1 \)) implies that for any \( \Pi \in \mathcal{P} \) there exist indices \( i \in \overline{N} \) yielding stable loops \((\Pi/K_i)'s\).

**Proposition 4**

Let \( \Theta \) be a compact set, and \( \theta \mapsto \Pi(\theta) \) continuous on \( \Theta \). Then, for any positive real \( \beta \), there always exists a finite model family \( \mathcal{M} \) such that

\[
\max_{\Pi \in \mathcal{P}} \min_{i \in S(\mathcal{P})} \|\Omega_{s/i}\|_{\text{ind}} =: \beta_{\text{RC}} \leq \beta
\] (35)

**Theorem 2**

Consider the ASC system \((\Pi/K_{\sigma(\cdot)}) \) (1), \( \Pi \in \mathcal{P}, \mathcal{P} \) compact, under zero initial conditions. Let \( \sigma(t) \) be selected in accordance with the HSL (8), with \( V_i(t) \) as in (28). Then, for any reference \( r \in \mathbb{S} \), the following properties hold:

(i) Let \( J_i(t) = J_i^{\text{RA}}(t) J_i^{\text{RC}}(t) \). Under a model distribution \( \mathcal{M}(\beta_{\text{RA}}), \beta_{\text{RA}} \) as in (31) \( \mathcal{M}(\beta_{\text{RC}}), \beta_{\text{RC}} \) as in (35) \}, the switching stops in finite time and \((\Pi/K_{\sigma(\cdot)}) \) is \( N \)-stable;

(ii) Under the same conditions as in (i), the total number of switches \( N_\sigma \) is bounded by \( N_\sigma \leq N[\beta_{\text{RA}}/h][N_\sigma \leq N[\beta_{\text{RC}}/h]], \) where \( \lfloor z \rfloor \) denotes the smallest integer greater than or equal to \( z \in \mathbb{R}_+ \); the occurrence of the condition \( J_i^{\text{RA}}(t) \geq \beta_{\text{RA}} J_i^{\text{RC}}(t) \geq \beta_{\text{RC}}, t \geq t_0, \forall \mathcal{P} \), ensures that no destabilizing controllers will be switched-on at and after time \( t \);

Finally, if \( h \geq \beta_{\text{RA}}[h \geq \beta_{\text{RC}}], \) each candidate controller can be switched-on no more than once, and any stabilizing controller with the \( \beta_{\text{RA}}[\beta_{\text{RC}}]-\)property, once switched-on in feedback with the plant, will be never switched-off thereafter.

**Proof**

See the Appendix.

### 3.3. Comparative assessment of test functionals

Next Table I depicts final comparative features of the test functionals considered in this paper. As elaborated hereafter, there are various reasons for assessing \( J_i^{\text{RC}} \) as the most effective functional among the possible alternatives.

- The reason for preferring \( J_i^{\text{RC}} \) over \( J_i^{\text{PA}} \) and \( J_i^{\text{RA}} \) is twofold. First, \( J_i^{\text{RC}} \) allows safe operation under all possible circumstances provided that solely Problem Feasibility hold. In particular, adoption of \( J_i^{\text{RC}} \) does not require that possible plant unstable modes be known. This is in contrast with the use of \( J_i^{\text{PA}} \) or \( J_i^{\text{RA}} \), which ask for the prior knowledge of the plant unstable modes. Additionally, \( J_i^{\text{RC}} \) does not need an explicit computation of the VR’s. In fact, as will be better clarified in the next section, computation of \( J_i^{\text{RC}} \) only hinges upon the mere knowledge of \( z \). Consequently, \( J_i^{\text{RC}} \) is on-line computable even when, as in the case of non-stably invertible controllers, \( v_i \) is not well-defined. On the contrary, the latter is required if \( J_i^{\text{PA}} \) or \( J_i^{\text{RA}} \) is used.

- The second comparison concerns \( J_i^{\text{RC}} \) and \( J_i^{\text{PC}} \). Notice that \( \Psi_{s/i} \) in (25) and \( \Omega_{s/i} \) in (33) satisfy

\[
\Omega_{s/i}(d) - \Psi_{s/i}(d) = \Omega_{s/i}(d) \Psi_{s/i}(d)
\]
The switching methodology developed in Section 3 hinges upon the computation of \( \tilde{z}_i \). In principle, \( \tilde{z}_i = z - z_{i,i} \) can be obtained by running the \( N \) tuned-loops (\( M_i/K_i \)) driven by \( v_i \).

### 4. \( J_i^{RC} \)-MMUASC IN A NOISY ENVIRONMENT

Having unveiled the advantages of MMUASC based on the \( J_i^{RC} \) identification functional (shortly, \( J_i^{RC} \)-MMUASC), the remaining part of this paper will be devoted to both analyze such a scheme in a noisy environment and exhibit efficient on-line computational architectures for general initial conditions. The main purpose, here, is to present the results when \( J_i^{RC} \) is computed via the \( \ell_2 \)-norm, the most frequent norm adopted in ASC systems [1, 17, 21]. For a more detailed treatment of the topics discussed hereafter (in particular, the extension to an arbitrary \( \mathcal{N} \)-norm), the reader is referred to the relevant literature [1, 23].

By assuming that the supervisor has only access to noisy corrupted plant I/O data, the switched system can be expressed as

\[
\begin{align*}
y(t) &= \Pi[u + n_u](t) + n_y(t) \\
u(t) &= K_{\sigma(t)}[r - y](t)
\end{align*}
\]

where \( n_u \) and \( n_y \) denote unknown disturbances. In order to analyze \( J_i^{RC} \)-MMUASC with reference to (37), relevant results for efficiently computing \( J_i^{RC} \) are first provided.

### Table I. Comparison among functionals \( J_i \)'s.

<table>
<thead>
<tr>
<th>Functionals</th>
<th>VR computation</th>
<th>Uncertainty</th>
<th>N-S conditions* for ( \mathcal{N} )-stability</th>
<th>N-S conditions* for the existence of ( \mathcal{M}(\beta) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( J_i^S )</td>
<td>Necessary</td>
<td>Non model-based</td>
<td>PF</td>
<td>Non model-based</td>
</tr>
<tr>
<td>( J_i^{PA} )</td>
<td>Necessary</td>
<td>Additive</td>
<td>( \beta_{PA} &lt; 1-h \Rightarrow PF )</td>
<td>( B1 )</td>
</tr>
<tr>
<td>( J_i^{PC} )</td>
<td>Unnecessary</td>
<td>Coprime factors</td>
<td>( \beta_{PC} &lt; 1-h \Rightarrow PF )</td>
<td>None</td>
</tr>
<tr>
<td>( J_i^{RA} )</td>
<td>Necessary</td>
<td>Additive</td>
<td>( \beta_{RA} &lt; \infty \Rightarrow PF )</td>
<td>( B1 )</td>
</tr>
<tr>
<td>( J_i^{RC} )</td>
<td>Unnecessary</td>
<td>Coprime factors</td>
<td>PF</td>
<td>None</td>
</tr>
</tbody>
</table>

N-S \( \equiv \) Necessary and Sufficient; PF \( \equiv \) Problem Feasibility.

*Assuming compactness of \( \Theta \) and continuity of the map \( \theta \mapsto \Pi(\theta) \).

Consequently, the following implications hold:

\[
\begin{align*}
\| \Psi_{s,i} \|_{\text{ind}} - \beta_{PC} &\Rightarrow \| \Omega_{s,i} \|_{\text{ind}} - \beta_{PC}(1 - \beta_{PC})^{-1} \\
\| \Omega_{s,i} \|_{\text{ind}} - \beta_{RC} &\Rightarrow \| \Psi_{s,i} \|_{\text{ind}} - \beta_{RC}(1 - \beta_{RC})^{-1}
\end{align*}
\] (36)

Equation (36) states that, in principle, any switching performance achievable via \( J_i^{RC} \) can also be obtained via \( J_i^{PC} \) and vice versa. Nonetheless, there are other reasons for considering \( J_i^{RC} \) preferable to \( J_i^{PC} \). Safe operation with \( J_i^{PC} \) requires the condition \( \beta_{PC} \leq 1-h \). As will be better clarified in next section, satisfaction of such a condition turns out to be a quite problematic task whenever the feedback loop is affected by unknown disturbances. In contrast with it, safe operation with \( J_i^{RC} \) is solely subject to Problem Feasibility.

- Despite the fact that asymptotic properties related to the use of either functionals rely on the same condition, viz. Problem Feasibility, there are two main reasons for preferring \( J_i^{RC} \) to \( J_i^S \). First, in accordance with Theorem 2, \( J_i^{RC} \) reduces the time duration of learning transients after start-up, which turns out to be typically quite long in UASC based on \( J_i^S \) [1]. Second, as already pointed out, \( J_i^{RC} \) can be used even when \( v_i \) is not well-defined, a situation the latter under which the adoption of \( J_i^S \) forces the designer to adopt variants of the basic scheme which imply further complications [26].
However, in order to avoid the computation of the VR’s, one can compute \( \tilde{z}_i \) via

\[
\tilde{z}_i(t) := (Q_{i/i})_0[\varepsilon_{i0}](t) = \left( \frac{1}{L_{i/i}(d)} \begin{bmatrix} -S_i(d)^	op & A_i(d)S_i(d) \\ B_i(d)R_i(d) & -A_i(d)R(d) \end{bmatrix} \right)_{0} \varepsilon_{i0}(t) \tag{38}
\]

\( \varepsilon_{i0} \) being the plant output prediction error based on the \( i \)th model \( M_i(d) = B_i(d)/A_i(d) \)

\[
\varepsilon_{i0}(t) := (L_i)_0[z](t) = A_i(d)y(t) - B_i(d)\delta u(t) \tag{39}
\]

computed as if the plant \( \Pi \) was in ‘zero initial conditions’ at time 0, \( \forall i \). As can be easily checked, under zero initial conditions and zero disturbances, (38) yields \( \tilde{z}_i = (\Omega_{s/i})_0[z_{i/i}], z_{i/i} = z - \tilde{z}_i \).

In the more general case of nonzero initial conditions and/or nonzero disturbances, it can be easily verified that (38) yields

\[
\tilde{z}_i = (\Omega_{s/i})_0[z_{i/i}] - (T_{s/i})_0[z] \tag{40}
\]

where \( T_{s/i} \) is the generalized sensitivity matrix of \((\Pi/K_i)\),

\[
T_{s/i}(d) := \frac{1}{L_{s/i}(d)} \begin{bmatrix} -B(d)S_i(d) & A(d)S_i(d) \\ B(d)R_i(d) & -A(d)R(d) \end{bmatrix} \tag{41}
\]

In turns, by letting \( \nu := [n_u \ n_y]' \), the rightmost term on the right-hand side of (40) can be divided into two parts. The first one, \( (T_{s/i})_0[v] := z_{i_i}' \), is due to disturbances. In particular, for indices related to stabilizing controllers, and under the assumption that \( n_u, n_y \in \ell_\infty(\mathbb{Z}_+) \), one has \( \|z_{i_i}'\|_2 < \gamma(t + 1) \), for some \( \gamma > 0 \). Conversely, the second term, \( (T_{s/i})_0[z - \nu] := \tilde{z}_{i_i}' \), arises from nonzero plant initial conditions. In addition, since \( z - \nu \) is the true plant I/O pair (Figure 5), one has \( (T_{s/i})_0[z - \nu](t) = 0 \), for every \( t \geq \max\{\deg B, \deg A\} \). Consequently, one has \( \|z_{i_i}'\|_2 < \xi \), for some \( \xi > 0 \), for indices related to stabilizing controllers. Summing up, if \( n_u, n_y \in \ell_\infty(\mathbb{Z}_+) \), one concludes that

\[
\|z_{i_i}'\|_2 \leq \|\Omega_{s/i}\|_\infty \|z_{i/i}'\|_2 + \gamma(t + 1) + \xi \quad \forall i \in S(\Pi) \tag{42}
\]

Based on (42), there are several alternatives for suitably modifying the test functional \( V_l \) in (28). A possible and effective way consists of replacing (28) by

\[
V^{RC}_l(t) := \begin{cases} \Lambda_l(t), & 0 \leq t \leq t_0 - 1 \\ \max_{0 \leq \ell \leq \tau} \|z_{i_i}'\|_2 \|s|_{t_0}^t + \|\eta_l|_{t_0}^t\|_2 & \text{elsewhere} \end{cases} \tag{43}
\]

where \( s|_{t_0}^t := [s(t_0), \ldots, s(t)] \), and \( \eta_l(t) := \max\{\mu, (z_{i_i}'(t))(t)\} \), \( \mu > 0 \). Further, \( \Lambda_l \) is a suitable functional to be used up to time \( t_0 - 1 \), \( t_0 \geq \max_{\Pi \in \mathcal{P}} \{\deg B, \deg A\} \).\footnote{In [1], the following choice is considered for \( \Lambda_l \):

\[
\Lambda_l(t) := \|z(t)\|_2^{1/2}, \quad i \in \mathbb{N} \tag{44}
\]

where \( \mathcal{T}_l \) is a constant depending on the magnitude of the mixed sensitivity of \((M_l/K_l)\).}

Some comments are in order. The role of \( \mu \) is to ensure that \( V^{RC}_l \) in (43) remains bounded for indices related to stabilizing controllers, hence to preserve the HSL applicability conditions. In this respect, it can be shown [23] that, under the only condition of Problem Feasibility, \( J^{RC}_{MMUASC} \) equipped with (43) ensures that for any plant initial state, reference \( r \in \mathbb{S} \) and bounded noises \( n_u \) and \( n_y \), the switching stops in finite time and \((\Pi/K_{\delta(i)})\) is \( \mathcal{P} \)-stable.

Conversely, the term \( z^{t_0 - 1} \) takes into account the effect, in the \( V_l^{RC} \)'s, of possible nonzero initial conditions. In this respect, since \( (T_{s/i})_0[z - \nu](t) = 0, \forall t \geq \max\{\deg B, \deg A\} \), it follows that \( z_{i_i}' \), though unknown, vanishes at an exponential rate as \( t \) increases. In other terms, the larger the \( t_0 \), the smaller the related \( \xi \). Accordingly, the introduction of \( t_0 \) (and, hence, \( \Lambda_l \)) in (43) is simply
Figure 5. Noisy ASC configuration.

Table II. Dependence of $\beta_{PC}$ and $\beta_{RC}$ from $N$.

<table>
<thead>
<tr>
<th>$N$</th>
<th>3</th>
<th>5</th>
<th>7</th>
<th>10</th>
<th>13</th>
<th>16</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_{PC}$</td>
<td>3.003</td>
<td>1.453</td>
<td>0.953</td>
<td>0.665</td>
<td>0.499</td>
<td>0.403</td>
</tr>
<tr>
<td>$\beta_{RC}$</td>
<td>9.474</td>
<td>3.248</td>
<td>1.507</td>
<td>0.812</td>
<td>0.556</td>
<td>0.419</td>
</tr>
</tbody>
</table>

instrumental for reducing the effect of $\xi$ in the final controller selection. As shown in detail in [1], effectiveness of $V_i^{RC}$ in the form (43) stems from the fact that one can suitably select $t_0$ so as to ensure that the resulting MMUASC system enjoys, under zero disturbances, properties similar to the ones stated in Theorem 2. In particular, it can be proved that there exists a large enough $t_0$ such that properties stated in Theorem 2 hold true over the time interval $Z_{t_0} := \{t_0, t_0 + 1, \ldots\}$ with $\beta_{RC}$ replaced by $(1 + \xi)\beta_{RC}$. For the necessary details, the reader is referred to [1].

4.1. A two-cart example

Simulation results for the proposed $J_i^{RC}$—MMUASC have already appeared in the literature. In particular, comparisons with the MMASC scheme of [13] and the UASC scheme of [17] are provided in [1]. Further, a detailed comparison between $J_i^{RC}$—MMUASC and the most recent adaptive control techniques [27, 28] is provided in [29]. In the sequel, simulation results comparing $J_i^{RC}$ and $J_i^{PC}$, both computed via the $\ell_2$-norm, are given.

Consider a discrete-time LTI plant, obtained from a continuous-time LTI plant by feeding its input via a zero order holder and sampling its output every 0.1 s (see [1] for more details).

The continuous-time LTI plant is made up of two carts mechanically coupled by a spring with uncertain stiffness $\gamma \in [0.25, 1.5]$ N/m. The problem, presented in [30] and discussed in [1, 31] within the context of ASC, is to control the position of one cart by applying a force to the other cart. $N$ different discrete-time controllers $K_i$ are designed relative to the nominal models $M_i$ corresponding to $N$ stiffness values logarithmically distributed (in view of the fact that the plant is harder to control for smaller values of $\gamma$) over the uncertainty interval of $\gamma$. At each time instant, the switched-on controller is chosen in accordance with (8). Simulation results reported hereafter refer to the following choices: the hysteresis is set equal to $h = 0.02$; $r$ is a square-wave of amplitude $\pm 2.5$ m with period 100 s.

Consider first the ideal case, viz. zero plant initial conditions, $n_u = n_y \equiv 0$, and $N = 16$ (Tables II and III). Accordingly, $V_i^{RC}$ is given by (28) with $J_i$ as in (34), which yields $\beta_{RC} = 0.419$. Similarly, $V_i^{PC}$ is given by (28) with $J_i$ as in (26), which yields $\beta_{PC} = 0.403$. As expected, the results reported in Figure 6 and Table IV show that, irrespective of $\gamma$, both functionals yield indistinguishable high performance, the stabilizing candidate controllers always being switched-on right after start-up.

The situation changes in case the validity conditions of Theorem 2 fail to hold. Assume that only three controllers (as reported in Tables II and V) are used. Accordingly, $\beta_{PC} = 3.003$. Assume also nonzero plant initial conditions and nonzero noises. More precisely, let the plant initial state $x(0)$ equal $x(0) = [1 - 0.5 - 0.060.7]'$, where the first and second components are the position and, respectively, velocity of the cart to which the force is applied; and the third and fourth components are the position and, respectively, velocity of the cart whose position is being controlled. Further
selection for test functional

Figure 6. Simulation results for $\gamma=0.5$, $N=16$, and $\sigma(0)=3$. (a) Tracking, plant input and controller selection for test functional $V_i$ in (28), with $J_i:=J_{RC}$ as in (34); (b) Tracking, plant input and controller selection for test functional $V_i$ in (28), with $J_i:=J_{PC}$ as in (26).

let $n_u$ and $n_y$ be Gaussian with zero mean and variance $\sigma^2=0.1$. Accordingly, $V_{iRC}$ is given by (43), here, with the choices $t_0:=70$ (corresponding to 7 s) and $\mu:=10^{-5}$ (in practice, $\mu$ never comes into play). Conversely, similar to (43),

$$V_{iPC}(t) := \begin{cases} A_i(t), & 0 \leq t \leq t_0 - 1 \\ \max_{0 \leq t_0 \leq t} \frac{\|z_i^0\|_2}{\|z^0\|_2} & \text{elsewhere} \end{cases}$$

with $t_0:=70$. Notice that, here, in contrast with $V_{iRC}$, $\mu$ can be set to zero as $V_{iPC}$ in (45) always takes on finite values irrespective of $n_u$ and $n_y$. Simulation results depicted in Figure 7 and Table VI show that $V_{iRC}$ yields, along with stability, performance similar to the one achieved in the ideal case. On the opposite, with the adoption of $V_{iPC}$, the resulting MMUASC system does not perform satisfactorily for any $\gamma$, and even becomes eventually stable for some values of $\gamma$. E.g., consider $\gamma=0.5$ (only $K_2$ stabilizes the plant). Since the model distribution is not dense enough and because of the presence of noises, $V_{iPC}$ takes on values larger than one. In turn, since $z \rightarrow \infty$ implies
In fact, in connection with that, among the various conceivable test functionals, the one deriving from $J$ was only focused on MMUASC equipped by reference loops adopted. Indeed, several issues remains open to investigation. The reference-loop identification architecture used in this paper allows one to address goals of robust stability and performance, at the expense of computational load and high performance must be tackled. In addition, the adopted approach only concerns LTI plants. However, a truly adaptive control methodology should enable the supervisor to handle also either slowly time-varying plants or infrequently abrupt plant changes.

Several issues remains open to investigation. The reference-loop identification architecture used in this paper allows one to address goals of robust stability and performance, at the expense of dense model distributions. In practice, trade-offs between conflicting requirements of a moderate memory/computational load and high performance must be tackled. In addition, the adopted approach only concerns LTI plants. However, a truly adaptive control methodology should enable the supervisor to handle also either slowly time-varying plants or infrequently abrupt plant changes.

In fact, in connection with $V^{PC}$, one can easily verify that, under nonzero initial conditions and/or nonzero disturbances, $\tilde{z}_i$ in (38) yields

$$\tilde{z}_i=(\Psi_{+\gamma})\theta(z)-(Q_{i\gamma}(B-A))\theta(z)$$

Consequently, in the presence of disturbances, the validity conditions of Theorem 1 need not hold since the rightmost term in the latter equation yields nonzero contribution.

### 5. CONCLUSIONS AND OPEN-PROBLEMS

A number of possible Multi-model based UASC systems were considered, each one equipped with a different test functional. Their features were comparatively analyzed mainly in terms of generality of applicability, stability inference and on-line implementability. The conclusion was that, among the various conceivable test functionals, the one deriving from $J_i^{RC}$ is the only one which exhibits the noticeable advantage of jointly enjoying the following properties: (i) It does not require computation of virtual references. Such a computation must be avoided in case of non-stably invertible controllers, and circumvented by rearranging the scheme at the possible cost of a resulting lower performance; (ii) It only requires computation of prediction errors, an advantageous feature from an implementational viewpoint; (iii) It only requires that the so-called Problem Feasibility be satisfied in order to yield stability. These conclusions add complementary motivations to the developments already elaborated in [1] whereby an in-depth study of MMUASC equipped by $J_i^{RC}$ with no indication at all on its advantage over conceivable alternative test functionals.

### Table IV. Comparison between $V_i^{RC}$ and $V_i^{PC}$ with $N = 16$ controllers in the ideal case (zero plant initial conditions, $n_u = 0$ and $n_y = 0$) with $\sigma(0) = 3$.

<table>
<thead>
<tr>
<th>$\gamma$</th>
<th>$\gamma = 0.3$</th>
<th>$\gamma = 0.5$</th>
<th>$\gamma = 0.7$</th>
<th>$\gamma = 0.9$</th>
<th>$\gamma = 1.1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\gamma$</td>
<td>2.68 [2.68]</td>
<td>2.72 [2.75]</td>
<td>2.72 [2.72]</td>
<td>2.55 [2.57]</td>
<td>2.58 [2.58]</td>
</tr>
<tr>
<td>Final switching instant with $V_i^{RC} [V_i^{PC}]$</td>
<td>0s [0s]</td>
<td>0.2s [0.24s]</td>
<td>0.12s [0.12s]</td>
<td>0.1s [0.1s]</td>
<td>0.1s [0.1s]</td>
</tr>
</tbody>
</table>

### Table V. Controllers coefficients for $N = 3$.

<table>
<thead>
<tr>
<th>$s_0$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$r_1$</th>
<th>$r_2$</th>
<th>$r_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_1$</td>
<td>43.4421</td>
<td>−128.9974</td>
<td>127.8710</td>
<td>−42.3118</td>
<td>−2.3876</td>
<td>1.9253</td>
</tr>
<tr>
<td>$C_2$</td>
<td>26.5098</td>
<td>−79.0064</td>
<td>78.7970</td>
<td>−26.2362</td>
<td>−2.3721</td>
<td>1.9023</td>
</tr>
<tr>
<td>$C_3$</td>
<td>13.2839</td>
<td>−40.3931</td>
<td>41.1406</td>
<td>−14.0267</td>
<td>−2.3223</td>
<td>1.8298</td>
</tr>
</tbody>
</table>

$\|\tilde{z}_i\|_2 / \|z\|_2 \rightarrow 1$, $K_3$ cannot be switched-off (Figure 8). Notice that, because of the presence of $n_u$ and $n_y$, with an adoption of $V^{PC}$, such phenomena may occur irrespective of the number of reference loops adopted.\textsuperscript{23}

\textsuperscript{23}In fact, in connection with $V^{PC}$, one can easily verify that, under nonzero initial conditions and/or nonzero disturbances, $\tilde{z}_i$ in (38) yields

$$\tilde{z}_i=(\Psi_{+\gamma})\theta(z)-(Q_{i\gamma}(B-A))\theta(z)$$

Consequently, in the presence of disturbances, the validity conditions of Theorem 1 need not hold since the rightmost term in the latter equation yields nonzero contribution.
Figure 7. Simulation results for $\gamma=0.5$, $N=3$ and $\sigma(0)=3$. (a) Tracking, plant input and controller selection for test functional $V_{i}^{\text{RC}}$ in (43) and (b) tracking, plant input and controller selection for test functional $V_{i}^{\text{PC}}$ in (45).

Table VI. Comparison between $V_{i}^{\text{RC}}$ and $V_{i}^{\text{PC}}$ with $N=3$ controllers in a real case (nonzero plant initial conditions, $n_u \neq 0$ and $n_y \neq 0$) with $\sigma(0)=3$.

| $\gamma$ | $\gamma=0.3$ | $\gamma=0.5$ | $\gamma=0.7$ | $\gamma=0.9$ | $\gamma=1.1$
|-----------|--------------|--------------|--------------|--------------|--------------|
| Maximum value of $|u|$ with $V_{i}^{\text{RC}}$ [$V_{i}^{\text{PC}}$] | 14.70 [$10^6$] | 8.08 [$10^6$] | 5.68 [2.64] | 2.79 [2.79] | 2.60 [2.60]
| Final switching instant with $V_{i}^{\text{RC}}$ [$V_{i}^{\text{PC}}$] | 6.1s [7.4s] | 8.6s [0s] | 20.5s [0s] | 0s [0s] | 0s [0s]

Figure 8. Simulation results for $\gamma=0.5$, $N=3$ and $\sigma(0)=3$: Time behavior of test functionals $V_{i}^{\text{PC}}$'s in (45) ($V_{1}^{\text{PC}}$ dotted line, $V_{2}^{\text{PC}}$ dashed line and $V_{3}^{\text{PC}}$ solid line).
Future studies must therefore be devoted to endow test functionals and switching logic having data-windowing or fading memory: To the best of the authors’ knowledge, an issue this, as yet, unsettled and deserving research efforts in the future.

APPENDIX

Proof of Theorem 1
We prove only (i), since (ii) is a direct consequence of (i). Further, we restrict attention to \( J^\text{PA} \) as the results for \( J^\text{PC} \) can be obtained along the same lines.

The switching stops in a finite time as a consequence of the following facts: first, each \( J^\text{PA} \) always takes on finite value whenever \( \|z\| > 0 \) (recall that \( A(0) = A_i(0) = 1 \) and \( B(0) = B_i(0) = 0 \), so that at the first time instant \( t \) where \( z(t) \neq 0 \) one has \( z(t) = [\delta u(t) \neq 0 y(t) = 0]' \), viz. \( \|n_{a/i}'\| > 0 \); in addition, each \( J^\text{PA} \) remain bounded over \( \mathbb{Z}_+ \) in view of stability of the tuned-loops \( (M_i/K_i)'s \). In turn, \( M(\beta_{\text{PA}}) \) ensures that \( J^\alpha_f(t) \leq \beta_{\text{PA}} + h < 1, t \in \mathbb{Z}_+, f \) being the final controller index. Thus, (20) implies

\[
\|z'\| - \|z_{f/f}'\| \leq \|z_{f/f}'\| \leq (\beta_{\text{PA}} + h)\|n_{a/i}'\| \leq (\beta_{\text{PA}} + h)(\|z'\| + \|v_f'\|),
\]

Notice that, regardless of the state of \( K_f \) is in at the switching final instant \( t_f \), one has

\[
S_f v_f(t) = R_f \delta u(t) + S_f y(t) = S_f r(t), \quad t \geq t_f + p
\]

with \( p := \max \{\deg S_f, \deg R_f \} \), viz. \( v_f \) converges exponentially to \( r \), and

\[
z_{f/f}(t) = (I_2 - Q_{f/f} L_f) z(t) = \frac{[A_f B_f]^T}{\lambda_{f/f}} S_f v_f(t) = \frac{[A_f B_f]^T}{\lambda_{f/f}} S_f r(t), \quad t \geq t_f + p
\]

where \( L_f = [-B_f A_f] \). Therefore, for some finite positive reals \( \kappa_1 \) and \( \kappa_2 \),

\[
\|z'\| \leq (1 - \beta_{\text{PA}} - h)^{-1} \|z_{f/f}'\| + \|v_f'\| \leq \kappa_1 \|r\| + \kappa_2, \quad t \in \mathbb{Z}_+
\]

where the last inequality follows from the fact that \( \lambda_{f/f} \) is strictly Schur.

Proof of Theorem 2
As beforehand, we prove only (i), since (ii) follows from (i). We also restrict attention to \( J^\text{RA} \) as the results for \( J^\text{RC} \) can be obtained along the same lines.

The switching stops in a finite time as a consequence of the following facts: first, each \( J^\text{RA} \) always takes on finite value whenever \( \|z\| > 0 \) (recall that \( A(0) = A_i(0) = 1 \) and \( B(0) = B_i(0) = 0 \), so that \( \Delta_i(0) = 0 \); At the first time instant \( t \) where \( z(t) \neq 0 \) one has \( z(t) = [\delta u(t) \neq 0 y(t) = 0]' \), viz. \( \|n_{a/i}'\| > 0 \), and, since \( A_i(0) = 0, n_{a/i}' = n_{a/i}' \); in addition, for every \( \Pi \in \mathcal{P} \), there are indices \( i \) for which \( J^\text{RA} \) remains bounded over \( \mathbb{Z}_+ \) as a consequence of Proposition 3, that is, \( J^\text{M}_i(t) \leq \beta_{\text{RA}}, t \in \mathbb{Z}_+, \) for some \( \beta_{\text{RA}} \in \mathbb{R}_+, f \) being the final controller index. Thus,

\[
\|z'\| - \|z_{f/f}'\| \leq \|z_{f/f}'\| \leq \beta_{\text{RA}} \|n_{f/f}'\| \leq \beta_{\text{RA}} (\|z_{f/f}'\| + \|v_f'\|)
\]

Again, regardless of the state of \( K_f \) is in at the switching final instant \( t_f \), one has

\[
S_f v_f(t) = R_f \delta u(t) + S_f y(t) = S_f r(t), \quad t \geq t_f + p
\]

with \( p := \max \{\deg S_f, \deg R_f \} \), viz. \( v_f \) converges exponentially to \( r \), and

\[
z_{f/f}(t) = (I_2 - Q_{f/f} L_f) z(t) = \frac{[A_f B_f]^T}{\lambda_{f/f}} S_f v_f(t) = \frac{[A_f B_f]^T}{\lambda_{f/f}} S_f r(t), \quad t \geq t_f + p
\]

where \( L_f = [-B_f A_f] \). Therefore, for some finite positive reals \( \kappa_1 \) and \( \kappa_2 \),

\[
\|z'\| \leq (1 + \beta_{\text{RA}}) \|z_{f/f}'\| + \beta_{\text{RA}} \|v_f'\| \leq \kappa_1 \|r\| + \kappa_2, \quad t \in \mathbb{Z}_+
\]

where the last inequality follows from the fact that \( \lambda_{f/f} \) is strictly Schur.
REFERENCES