9

Estimation of the Decomposition Formula

9.1 Introduction

The direct vs. compositional decomposition in equation (6.4) is for continuous changes over time. Nevertheless, demographic data are only available for stocks at a particular point in time or flows over some time period or categorical values for groups. The same accounts for many of the equations presented in Part III. However, these equations can be estimated using the available data and assuming some change over time in the demographic variables under study. The estimations are very precise if these assumptions follow the observed change. However, in some applications the estimations might carry big residual terms, which is a clear sign of wrong assumptions about the change over time of the demographic variables.

This chapter introduces several estimation procedures for the formulas pertaining to direct vs. compositional decomposition. The first section reviews the advantages of visualizing demographic data as vectors and matrices. Rewriting the data in this compact way facilitates the calculations of our main equation. The remaining section is concerned with ways of estimating derivatives over time for demographic measures. Here, we show the estimation procedures used for the applications presented in the book on changes in mortality, fertility and growth rate measures.

to examine the effects of a change in age-specific rates on the various lifetable functions. This analytical approach is designated as sensitivity analysis. Differentiating a matrix is the procedure of finding partial derivatives of the elements in a matrix function with respect to an argument of the matrix or a scalar. We focus on changes over time by looking at the partial derivatives of the entries in the matrix with respect to time. For example, the change over time of the vector $V(t)$ is

$$\frac{\partial}{\partial t}[V] = \left[\ \tfrac{\partial}{\partial t}v(1,t), \quad \tfrac{\partial}{\partial t}v(2,t), \quad \cdots, \quad \tfrac{\partial}{\partial t}v(\omega,t)\ \right]. \tag{9.4}$$

Similar to the derivative of a product of functions, one can calculate the derivative of a product of vectors. For example, the derivative of the vectors V and W^T is equal to

$$\frac{\partial}{\partial t}\left[VW^T\right] = \frac{\partial}{\partial t}[V]\,W^T + V\frac{\partial}{\partial t}\left[W^T\right]. \tag{9.5}$$

This property is easily generalized to the case of three vectors. It is therefore possible to calculate the derivative of an average as the derivative of the product of the three terms in (9.3),

$$\frac{\partial}{\partial t}\bar{v} = \frac{\partial}{\partial t}\left[VW^T\left(W1^T\right)^{-1}\right]$$

$$= \frac{\partial}{\partial t}[V]\,W^T\left(W1^T\right)^{-1} + V\frac{\partial}{\partial t}\left[W^T\right]\left(W1^T\right)^{-1} + VW^T\frac{\partial}{\partial t}\left[\left(W1^T\right)^{-1}\right]. \tag{9.6}$$

Recalling the notation of the dot over a variable to denote the derivative with respect to time, equation (9.6) can also be written as

$$\dot{\bar{v}} = \dot{V}W^T\left(W1^T\right)^{-1} + V\dot{W}^T\left(W1^T\right)^{-1} - \left(VW^T\left(W1^T\right)^{-2}\right)\left(\dot{W}1^T\right),$$

where the derivative of the last terms is $\frac{\partial}{\partial t}\left[\left(W1^T\right)^{-1}\right] = -\left(W1^T\right)^{-2}\left(\dot{W}1^T\right)$. Let the intensity of the weighting function be

$$\acute{W}(t) = \left[\ \frac{\frac{\partial}{\partial t}w(1,t)}{w(1,t)}, \quad \frac{\frac{\partial}{\partial t}w(2,t)}{w(2,t)}, \quad \cdots, \quad \frac{\frac{\partial}{\partial t}w(\omega,t)}{w(\omega,t)}\ \right]. \tag{9.7}$$

If we rewrite the derivative of a vector as the product of the relative derivative of the vector by the vector, $\dot{W}1^T = \acute{W}W^T$, we obtain the familiar result of equation (6.4),

$$\dot{\bar{v}} = \dot{V}W^T\left(W1^T\right)^{-1} + V\dot{W}^T\left(W1^T\right)^{-1} - \left[VW^T\left(W1^T\right)^{-1}\right]\left[\left(\acute{W}W^T\right)\left(W1^T\right)^{-1}\right] \tag{9.8}$$

or expressed in the notation used in equation (6.4)

$$\begin{aligned}\dot{\bar{v}} &= \bar{\dot{v}} + \overline{v\acute{w}} - (\bar{v})\left(\bar{\acute{w}}\right)\\ &= \bar{\dot{v}} + C(v,\acute{w}).\end{aligned} \tag{9.9}$$

The relevance of this formulation in vector form becomes apparent in cases where we have numerous types of compositions in a population. As shown in Chapter 8, the decomposition formula can be further decomposed into numerous covariance terms. Each of the covariances accounts for one composition of the population. The initial data in a cross-way contingency table can be changed into vectors of the type of $V(t)$ and $W(t)$ as presented in (9.1) and (9.2). For example, if we wish to study the crude death rate of selected European countries we divide the population into age groups and countries. In the applications presented in Chapter 8 we examined 12 age groups and 14 countries. The death rate at age a and for country c is $m_{ac}(t)$, and the matrix $U^*(t)$ of death rates is

$$U^*(t) = \begin{bmatrix} m_{1,1}(t) & m_{1,2}(t) & \cdots & m_{1,12}(t) \\ m_{2,1}(t) & \ddots & & \\ \vdots & & & \\ m_{14,1}(t) & \cdots & & m_{14,12}(t) \end{bmatrix}. \tag{9.10}$$

Similarly the matrix of the population sizes $N^*(t)$ is defined by cells for the population size by age groups and countries $N_{ac}(t)$. Following the rearrangements of matrices into vectors shown by Willekens (1977), we can transform $U^*(t)$ and $N^*(t)$ into the vectors $U(t)$ and $N(t)$. Let the first 12 cells in the vector $U(t)$ be the first row in $U^*(t)$, the next 12 cells of $U(t)$ be the second row of $U^*(t)$, and similarly for the rest of the rows in $U^*(t)$. The new vector $U(t)$ is

$$U(t) = \left[\ \{m_{1,\bullet}(t)\},\ \ \{m_{2,\bullet}(t)\},\ \ \cdots\ \ \{m_{14,\bullet}(t)\}\ \right]$$

$$= \left[\ m_{1,1}(t),\ \ m_{1,2}(t),\ \ \cdots\ \ m_{1,12}(t),\ \ m_{2,1}(t),\ \ m_{2,2}(t),\ \ \cdots\ \ m_{14,1}(t),\ \ \cdots\ \ m_{14,12}(t)\ \right].$$

Likewise the population size matrix, $N^*(t)$, can be transformed into the vector $N(t)$. By letting $V(t) = U(t)$ and $W(t) = N(t)$, we can conclude that all the developments shown in this section are valid for these vectors. Equation (9.8) is then expressed as

$$\dot{\bar{d}} = \dot{U}N^T\left(N1^T\right)^{-1} + U\dot{N}^T\left(N1^T\right)^{-1} - \left[UN^T\left(N1^T\right)^{-1}\right]\left[\left(\acute{N}N^T\right)\left(N1^T\right)^{-1}\right], \tag{9.11}$$

which are the explicit calculations required to obtain the decomposition. The only two vectors that correspond to changes are $\dot{U}$ and $\acute{N}$. These changes occur during the entire period under study, from time t to $t+h$. In all the applications shown in the book we have chosen the mid-point as the moment for representing this change, $t+h/2$. The other terms of (9.11) are functions of the vectors $\dot{U}$ and $\acute{N}$ that need to be allocated at time $t+h/2$. Once the four vectors, $\dot{U}(t+h/2)$, $\acute{N}(t+h/2)$, $U(t+h/2)$ and $N(t+h/2)$, are obtained at time $t+h/2$ it is simple a matter of mechanically putting them in the order specified in equation (9.11). This is true for any application that involves equation (9.8).

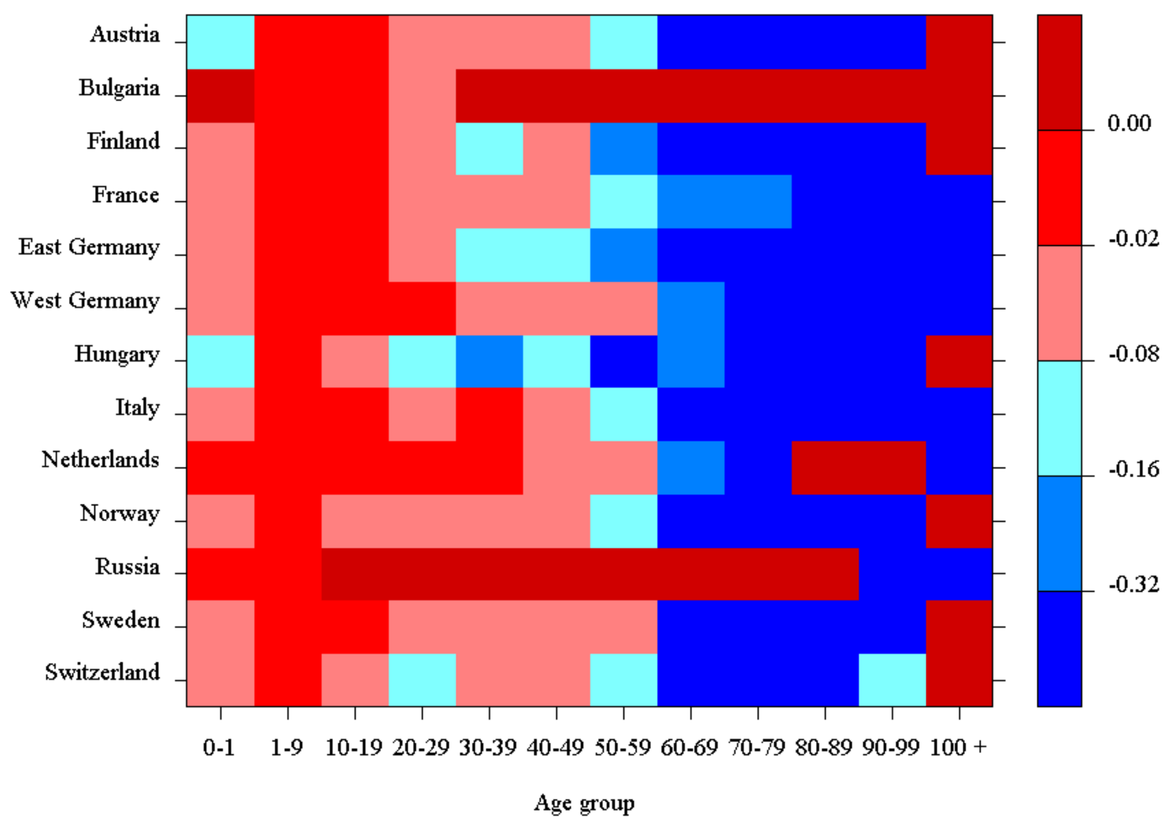
Chapter 8 illustrates how each age group and country contribute to the total change in the crude death rate by using a lexis surface. Depicting the matrices $\dot{U}^*(t+h/2)$ and $\acute{N}^*(t+h/2)$ by using a lexis surface yields some interesting insights. These matrices account for the change over time in the death rates $\dot{U}(t+h/2)$ and in the population size $\acute{N}(t+h/2)$, only in matrix format. The $\acute{N}^*(t+h/2)$ matrix can also be interpreted as a matrix of growth rates, where each cell is equal to $\acute{N}^*_{a,c}(t+h/2) = r_{a,c}(t+h/2)$.

9.2.3 Matrices of Change

Figures 8.1 and 8.2 show the age and country decomposition of the change of CDR into the direct effect and the compositional effect. Each square in these diagrams corresponds to the contribution of one age group in a specific country to the total change in the crude death rate.

Figure 9.1 displays a lexis surface of the change over time in the matrix of death rates $\dot{U}^{*}(t + h/2)$. The change in the matrix of deaths is represented in the rows by countries and in the columns by age groups. In Figure 8.1 France, East and West Germany, Hungary and

Figure 9.1: Lexis surface of change in the death rates by age groups and selected European countries from 1992 to 1996.



Italy were the main contributors to the average decline in mortality rates. At the same time, it can be seen that other countries, such as Bulgaria, Russia and the Netherlands experienced fewer improvements in their survivorship profile. Although, Figure 9.1 shows similar results, it is difficult to draw conclusions on how a particular country and age group affect the total outcome of the overall crude death rate of these countries. For example, in Figure 8.1 we see that several age groups in Italy contribute to the decline of the CDR, while in Norway few age groups are involved in the increase of this measure. In Figure 9.1 we find a greater number of age groups in Norway than in Italy in which improvements in mortality took place.

In Figure 8.1, the improvements in mortality rates are concentrated in the age groups 40 to 99. At the same time the young and the very old age groups countered the decrease in mortality. Figure 9.1 shows that the improvements in mortality mainly occur between the age groups 60 to 99.

We conclude here that Figure 8.1 is more informative in depicting direct effect in the study of the CDR than $\dot{U}^*(t + h/2)$. Similar conclusions would be reached for other demographic measures decomposed in this book. The direct component is a measure of the change in the variable of interest weighted by its relevance in the population.

9.3 Estimation of Derivatives in Demography

Both direct vs. compositional decomposition and the other formulas for change over time of demographic averages are exact. The implicit assumption is that these functions are differentiable with respect to time. This also implies that the averages are continuous over time. Continuity in itself may not be a problem, but the fact that demographic data are found in chunks poses a problem. It is, therefore, necessary to estimate precise formulas with the available data on the variables under study. However the estimation will contain minor inaccuracies in some of the tables presented above. For example, for the estimation of some formulas we substituted sums for integrals, and the demographic average was calculated as

$$\begin{aligned} \bar{v}(t) &= \frac{\int_0^\omega v(a,t)w(a,t)da}{\int_0^\omega w(a,t)da} \approx \frac{\sum_{a=0}^{\omega} v(a+0.5,t)w(a+0.5,t)}{\sum_{a=0}^{\omega} w(a+0.5,t)} \\ &= \frac{v(0.5,t)w(0.5,t) + ... + v(\omega-0.5,t)w(\omega-0.5,t)}{w(0.5,t) + ... + w(\omega-0.5,t)}. \end{aligned} \tag{9.12}$$

In this section we show which assumptions were made for calculating the changes over time in the applications presented above. The next subsection explains the different options for estimating derivatives and relative derivatives of functions that are only known in discrete moments. Successively, we illustrate this by case studies of changes in mortality, fertility and growth measures.

9.3.1 Changes over Time

This section presents the estimation for the change over time of demographic variables used in direct vs. compositional decomposition. These estimations depend on the assumptions of the type of change. Therefore, here we present the two types of change that we have used for our applications.

Change in the population is expressed as rate of increase or growth rate. This notion of change can be generalized to other demographic variables. We begin this section by presenting the relationship between a population at two points in time.

Let the population change be denoted as the difference of the population size at two points in time t and $t + h$, $\Delta N(t) = N(t + h) - N(t)$. When this time interval is small enough the crude growth rate is the ratio of the difference $\Delta N(t)$ divided by the person-years lived over the period, approximated by $N(t)h$. The limit of this ratio when h approaches 0 is the instantaneous growth rate at time t

$$r(t) = \lim_{h \to 0} \frac{N(t+h) - N(t)}{N(t)h} = \frac{\dot{N}(t)}{N(t)}, \tag{9.13}$$

where the second equality holds because this limit is equal to the derivative of $N(t)$. Equation (9.13) can be rewritten as the intensity or derivative of the logarithm of $N(t)$,

$$r(t) = \acute{N}(t) = \frac{d \ln [N(t)]}{dt}. \tag{9.14}$$

Based on equation (9.13) a new expression for population change over time can be developed. By integrating both sides of (9.14) from t to $t+h$ we obtain the desired relation

$$\int_t^{t+h} r(x)dx = \ln [N(x)] \Big|_t^{t+h} = \ln \left[\frac{N(t+h)}{N(t)} \right]. \tag{9.15}$$

This expression for the instantaneous population growth rate at time t can also be used to derive changes from other demographic variables.

Exponential Change

When data are available for time t and $t+h$, we generally used the following approximations for the value at the mid-point $t+h/2$. Variables that have a constant positive growth during the interval are considered to be growing exponentially. We can conclude from (9.15) that the growth for those variables is $\int_t^{t+h} r dx = rh$. This accounts for population size, births and others. Analogous to $r(t) = \acute{N}(t)$, for the relative derivative of the function $v(a,t)$ we have

$$\acute{v}(a, t+h/2) \approx \frac{\ln \left[\frac{v(a,t+h)}{v(a,t)} \right]}{h}. \tag{9.16}$$

Dividing by the number of years in the period h implies that the relative change in $\acute{v}(t+h/2)$ is annualized. The value of the function at the mid-point $v\,(a, t+h/2)$ is estimated by

$$v(a, t+h/2) \approx v(a,t) e^{(h/2)\acute{v}(a,t+h/2)}. \tag{9.17}$$

Substituting the right-hand side of (9.16) by $\acute{v}(a, t+h/2)$ in (9.17) yields the equivalent approximation

$$v(a, t+h/2) \approx [v(a,t) v(a,t+h)]^{1/2}. \tag{9.18}$$

This is a standard approximation in demography, as shown in Preston et al. (2001).

The derivative of the function $v(a, t+h/2)$ is estimated from the previous two equations (9.16) and (9.18) by

$$\dot{v}(a, t+h/2) = \acute{v}(a, t+h/2) v(a, t+h/2). \tag{9.19}$$

The equations (9.16), (9.18) and (9.19) are used when the change of the variable under study follows a constant positive growth. For example, the rate of progress in reducing the death rate $\rho(a, t+h/2)$, which was used in Table 4.8 and Figures 6.1 and 6.2, was calculated at time $t+h/2$ as

$$\rho(a, t+h/2) = -\acute{\mu}(a, t+h/2) \approx -\frac{\ln \left[\frac{\mu(a,t+h)}{\mu(a,t)} \right]}{h}. \tag{9.20}$$

Linear Change

The most common way to compute derivatives in numerical analysis is to use finite difference methods (see Judd (1988)). If the variable of interest, for instance $\bar{v}(t)$, has a linear growth, then the approximation of the derivative $\dot{\bar{v}}(t+h/2)$ is the one-sided finite difference formula

$$\dot{\bar{v}}(t+h/2) \approx \frac{\bar{v}(t+h) - \bar{v}(t)}{h}, \tag{9.21}$$

where h is the appropriate step size chosen, which normally is between one and ten years. When we divide by the number of years in the period this implies that the change in the average, $\dot{\bar{v}}(t+h/2)$, is annualized.

For a more general case, if the variable of interest is a function of several variables, such as the age-specific variable $v(a,t)$ at age a and time t, then the one-sided finite difference formula that substitutes the partial derivative with respect to time is

$$\dot{v}(a,t+h/2) = \frac{\partial v(a,t)}{\partial t} \approx \frac{v(a,t+h) - v(a,t)}{h}. \tag{9.22}$$

For example, in Table 7.2 we studied the change over time of the population growth rate of country i, denoted as $r_i(t)$. We assumed that the change is linear

$$\dot{r}_i(t+h/2) = \frac{\partial r_i(t)}{\partial t} \approx \frac{r_i(t+h) - r_i(t)}{h}. \tag{9.23}$$

The value of the function $v(a,t)$ at the mid-point $v\left(a,t+h/2\right)$ is estimated by

$$v(a,t+h/2) \approx \frac{v(a,t+h) + v(a,t)}{2}. \tag{9.24}$$

The following application contrasts the possible errors due to different assumptions of the change over time. Table 9.1 presents the decomposition of the change over time in the crude death rate of the United States. The age-specific death rates are assumed to change linearly and exponentially over time. In both cases the population size is growing exponentially, therefore by contrasting these two columns we observe only the change due to the age-specific death rates. In all the applications presented in the book only two points of the demographic variables over time are known. The straight line between these points is always above the exponential curve between these points. As a result assuming an exponential growth in the death rates implies lower values for the estimated mid-year crude death rate, direct change and compositional component. In Table 9.1 these results are shown in the rows of $d(1990)$, $\bar{\dot{\mu}}$ and $C(\mu, r)$ respectively. However, the estimated change $\dot{d}$ is the same for both assumptions.

Other types of growth rates could be used for estimating the formulas. One example is the s-shaped curve, like the logistic formula. Nevertheless, in the applications presented in this book we only used exponential and linear growth to estimate derivatives.

9.3.2 Mortality Measures

White (2002) has shown that mortality measures have followed linear patterns over the last fifty years. White considered a simple regression model for each mortality measure. Letting

Table 9.1: Crude death rate, $d(t)$, per thousand, and decomposition of the annual change over time from 1985 to 1995 for the United States. The values in the columns are derived from the assumption that the age-specific death rates change exponentially and linearly over time.

United States	linear growth	exponential growth
$d(1990)$	8.954	8.762
$d(1985)$	9.683	9.683
$d(1995)$	7.941	7.941
$\dot{d}(1990)$	-0.174	-0.174
$\bar{\acute{\mu}}$	-0.249	-0.245
$C(\mu, r)$	0.075	0.071
$\dot{d} = \bar{\acute{\mu}} + C(\mu, r)$	-0.174	-0.174

Source: Author's calculations described in Chapter 9, based on the United Nations Data Base (2001).

k_1 and k_2 be constants the model implemented is

$$mortality\ measure\ in\ year\ t = k_1 + k_2 * year\ t. \tag{9.25}$$

Then, life expectancy and logged age-specific death rates were tested for goodness of fit by models of the type illustrated by (9.25) in 21 developed countries. The averages over countries of straight lines fitting the mortality measures, R^2 were above 0.91 for all the measures. Considering that 1 is the perfect line, these are very good fits. Life expectancy proved to be the most linear of all these measures. It therefore seems appropriate to focus on linear trends when analyzing life expectancy and logged age-specific death rates.

In Tables 6.2 and 6.7 we assumed that life expectancy changed over time following a linear trend as in (9.21)

$$\dot{e}^o(0, t + h/2) = \frac{\partial \dot{e}^o(0, t)}{\partial t} \approx \frac{\dot{e}^o(0, t + h) - \dot{e}^o(0, t)}{h}. \tag{9.26}$$

In Tables 6.1, 6.3, 7.1, 8.2, 8.3 and 8.5 the derivative of the force of mortality (or age-specific death rates) at time $t + h/2$ is estimated using the exponential change in (9.19),

$$\dot{\mu}(a, t + h/2) = \acute{\mu}(a, t + h/2)\mu(a, t + h/2), \tag{9.27}$$

where the force of mortality at time $t + h/2$ is estimated using (9.18)

$$\mu(a, t + h/2) \approx [\mu(a, t)\mu(a, t + h)]^{1/2} \tag{9.28}$$

and the intensity is estimated as (9.16)

$$\acute{\mu}(a, t + h/2) \approx \frac{\ln\left[\frac{\mu(a,t+h)}{\mu(a,t)}\right]}{h}. \tag{9.29}$$

Also the force of mortality is defined as a change. $\mu(a,t)$ is equal to the negative intensity of the survival function $\ell(a,t)$ over age. When data were available for ages a and $a+k$ we used the following approximation for the force of mortality at age $a+k/2$

$$\mu(a+k/2,t) \approx \int_a^{a+k} \mu(x,t)dx = -\frac{\ln\left[\frac{\ell(a+k,t)}{\ell(a,t)}\right]}{k}. \tag{9.30}$$

The force of mortality is normally defined for each age, which means that k would then be 1. equation (9.30) was used in Table 4.8 and Figures 6.1 and 6.2 to estimate the force of mortality.

For Table 4.8 and Figures 6.1 and 6.2 it was necessary to calculate the other functions involved in the decomposition at that age, because the force of mortality in (9.30) is at age $a+k/2$. The survival function $\ell(a,t)$ and the remaining life expectancy $e^o(a,t)$ at age $a+k/2$ were calculated using a formula analogous to (9.18). The lifetable distribution of deaths was calculated as

$$f(a+k/2,t) \approx \mu(a+k/2,t)\ell(a+k/2,t). \tag{9.31}$$

9.3.3 Fertility Measures

Smith et al. (1996) studied the trends in fertility of unmarried women in the United States using a decomposition technique proposed by Das Gupta (1993). They found that between 1960 and 1992 the fertility of Black Americans outside wedlock underwent an exponential growth. Nevertheless, the decomposition technique used does not allow exponential growth.

The formulas presented in Part III allow for different types of change. Fertility changes on a yearly basis can occur at any age, birth order and because of quantum and tempo effects. Here, we took changes over time in births rates by age into account without looking at other types of change. The other possible reasons for change in the fertility rates are beyond the focus of this book. However, decomposition methods are certainly an interesting tool for examining other types of changes in fertility measures.

Derivatives and intensities of fertility measures as shown in Tables 4.6, 6.5, 6.6 and 8.6 are estimated by using exponential growth in (9.16) to (9.19). For example, in Table 6.5 the intensity with respect to time of the birth rates among married women $\acute{b}_m$ was calculated as

$$\acute{b}_m(a,t+h/2) \approx \frac{\ln\left[\frac{b_m(a,t+h)}{b_m(a,t)}\right]}{h}. \tag{9.32}$$

9.3.4 Growth Measures

In Chapter 4 we presented growth rate measures based on several researchers, among them the studies by Preston and Coale (1982) and Arthur and Vaupel (1984). Their work presented generalizations of the stable population model and some relationships that hold among demographic variables. Following these efforts to understand the population growth, Kim (1986) derived the formulas for discrete time and age. We present some of these formulations here.

The population growth rates are calculated as shown in equation (9.16). If the population size was known both at time t and $t+h$ for age a we get

$$r(a,t+h/2) = \acute{N}(a,t+h/2) \approx \frac{\ln\left[\frac{N(a,t+h)}{N(a,t)}\right]}{h}. \tag{9.33}$$

This approximation was applied in all tables that involved a population growth rate in Chapters 4, 6, 7 and 8.

Tables 4.9, 7.2 and 7.3 present the change over time of growth rates. In these examples we assume that the population growth rate changed linearly and we use differences to estimate the change over time

$$\dot{r}_i(t+h/2) = \frac{\partial r_i(t)}{\partial t} \approx \frac{r_i(t+h)-r_i(t)}{h}. \tag{9.34}$$

In Table 4.11 two measures of the population growth rate are shown. The first corresponds to the population growth rate calculated by using equation (9.33) for 1990

$$r(1990) = \acute{N}(1990) = \frac{\ln\left[\frac{N(1995)}{N(1985)}\right]}{10} = 0.506. \tag{9.35}$$

The second measure corresponds to the growth rate calculated as an average of age-specific growth rates, estimated from equation (9.16), weighted by the population size at each age

$$\begin{aligned} r(1990)\ * &= \frac{\int_0^\omega r(a,1990)N(a,1990)da}{\int_0^\omega N(a,1990)da} \\ &= \frac{\int_0^\omega \frac{\ln\left[\frac{N(a,1995)}{N(a,1985)}\right]}{10} N(a,1990)da}{\int_0^\omega N(a,1990)da} = 0.504. \end{aligned} \tag{9.36}$$

In the same Table 4.11 we see that the estimated decomposition corresponds exactly to the value of equation (9.36).

The reason for the difference between (9.35) and (9.36) when estimating population growth rates is due to the use of logarithms in equation (9.16), that is, $\ln[a+b] \neq \ln[a] + \ln[b]$.

For the applications that use exponential growth it is possible to reduce the bias in estimation procedures by redefining the observed changes in the demographic average $\dot{\bar{v}}(t)$. The observed change over time in a demographic variable is calculated as

$$\dot{\bar{v}} = \frac{\partial}{\partial t}\left[\frac{\int_0^\infty v(x,t)w(x,t)dx}{\int_0^\infty w(x,t)dx}\right] = \int_0^\infty \frac{\partial}{\partial t}\left[\frac{v(x,t)w(x,t)}{\int_0^\infty w(x,t)dx}\right] dx. \tag{9.37}$$

Using the right-hand side of (9.37) reduces considerably the possible bias due to the logarithm function, $\ln[a+b] \neq \ln[a] + \ln[b]$. The right-hand side of (9.37) was implemented in Table 4.11 in order to obtain precise results.

9.2 Vector Formulation of Direct vs. Compositional Decomposition

9.2.1 Matrix Notation in Demography

Rewriting demographic data in matrix notation has gradually been established within projection theory since the contribution of Leslie (1945) (in Smith and Keyfitz (1977)). Arranging the data in this way often facilitates computer applications and allows us to establish relations in population dynamics through matrix algebra.

The matrix theory has also been used in another field of demography, namely in the development of multi-state or increment-decrement lifetables. Rogers developed multi-state demography, which simultaneously analyzes the spatial dynamics of a system with several interdependent states that are linked by transitions. By letting M be a matrix, each cell in the matrix m_{ij} represents the transitions from state i to state j. Schoen (1988), Willekens et al. (1993) and Rogers (1995) made further developments in multi-state demography. An example of this state space is shown by Willekens et al. (1993) using four marital statuses and transitions between these states.

The matrix formulation is very important for several areas of demography. Therefore, as a first step, we introduce our main formula in vector terms in this chapter.

9.2.2 The Direct vs. Compositional Decomposition

This subsection introduces the key equation (6.4) in vector terms, considered as a matrix of one by ω-columns. Let $V(t)$ and $W(t)$ be two vectors of ω-entries at time t. The entries of $V(t)$ are $\{v(x,t)\}$ representing the different values of the characteristic x. If x is age a then each cell $v(a,t)$ corresponds to the values of the demographic variable of interest in the age group a to $a+1$ at time t,

$$V(t) = \left[\begin{array}{cccc} v(1,t), & v(2,t), & \cdots, & v(\omega,t) \end{array}\right]. \tag{9.1}$$

Similarly the weighting function $w(x,t)$ can be arranged in $W(t)$

$$W(t) = \left[\begin{array}{cccc} w(1,t), & w(2,t), & \cdots, & w(\omega,t) \end{array}\right]. \tag{9.2}$$

The mathematical expectation shown in equation (2.1) can now be written in matrix notation as

$$\bar{v}(t) = VW^T \left(W1^T\right)^{-1}, \tag{9.3}$$

where W^T is the transpose of W and 1 represents a vector with ones in all the entries. The product $W1^T$ is a fixed number and it can also be considered as a matrix of one by one. The notation $\left(W1^T\right)^{-1}$ denotes the inverse of $W1^T$. This means that the product of the two gives us a value of one.

Equation (6.4) can now be expressed in vector notation following the derivative of a product of vectors in equation (9.3). Willekens (1977) used the matrix differentiation techniques

9.4 Conclusion

An interesting debate has arisen in studies of both fertility and mortality concerning cohort and period changes. Bongaarts and Feeney (1998) and (2002) discussed about whether to use adjusted measures of fertility and mortality to understand observed changes. A similar context is found when applying the equations presented in Part III. Our equations are exact, and the techniques are applicable to many circumstances. Nevertheless, it is necessary to look at the particular period under study to understand which kind of assumptions should be made for the changes over time.

In this final chapter we have shown how direct vs. compositional decomposition can be estimated. First, a matrix formulation (9.8) of the main equation was introduced followed by an application of the matrices under study. This formulation also points out which calculations are explicitly required to obtain the decomposition. It involves two vectors that correspond to changes occurring during the period under study. In all the applications of direct vs. compositional decomposition shown in the book we have chosen the mid-point as the moment for representing this change. The other vectors are allocated at that moment, so it is a simple mechanical procedure to apply them in the order specified by equation (9.8). This section also allowed us to re-assess the relevance of the components of change, and of the direct and compositional effects in our decomposition.

This chapter was dedicated to the estimation of derivatives and intensities over time. Two types of change are suggested, but many others could be applied because the formulas are flexible and can be applied to any assumption of the trends over time in the demographic variable under study. We also gave some suggestions on how to reduce the bias in estimation procedures.