8

Multidimensional Decompositions

8.1 Introduction

In Chapter 6, where the decomposition formula (6.4) was introduced, several applications are shown. These illustrations look at the change over time in demographic variables accounting for the change in the variable of interest and for the compositional effect. Two types of compositions of the population are separately studied, the age structure of the population and the composition of the population by country of residence.

Sometimes several compositions simultaneously affect the change in the average under study. In these situations, it is necessary to have an extension of equation (6.4) to decompose the change over time of the demographic variable. This chapter extends the decomposition (6.4) to the case of numerous dimensions of population heterogeneity contained within the same formula.

In Chapter 3 we presented Kitagawa's decomposition formula which was later extended for situations involving numerous compositions. The methodologies of Cho and Retherford (1973), Kim and Strobino (1984) and Das Gupta (1978) have been used in demography as extensions. A similar approach, suggested by Oosterhaven and Van der Linden (1997) for economic variables is the structural decomposition. These extensions inspired our present extension of equation (6.4).

The total crude death rate, CDR, of selected European countries is calculated as the average of the countries' crude death rates weighted by the population size. There are three components influencing the change of this CDR: changes in the age- and country-specific death rates, changes in the age structure of each country and changes in the population distribution over countries. This chapter shows how to perform an extension of equation (6.4) when demographic variables include numerous structures of the population.

Two ways of solving this problem are shown here. The first method follows Cho and

Retherford's (1973) idea of applying the decomposition formula twice. This part also refers to the work by Kim and Strobino (1984) where some hierarchy among the compositional components is imposed.

The second version is related to the proposals made by Das Gupta (1978) and Clogg's (1978) methodology. Das Gupta's (1978) work is used for separating the weighting function into parts corresponding to each compositional component. Another option for this extension is found by using the Clogg's (1978) methodology of "purging" the undesired factors. In these approaches the different terms account for the different compositional components studied.

8.2 Averages of Averages

8.2.1 Average Age of the Population

Suppose a population consists of a number of subpopulations. Let the average age in each subpopulation i be $\bar{a}_i(t)$,

$$\bar{a}_i(t) = \frac{\sum_{a=0}^{\omega} a n_{ia}(t)}{\sum_{a=0}^{\omega} n_{ia}(t)}, \tag{8.1}$$

where $n_{ia}(t)$ is the age-specific subpopulation size. Let the age-specific growth rates be $r_{ia}(t)$, the total subpopulation size $N_i(t)$, and the overall growth rates $\bar{r}_i(t)$. The average age of the entire population $\tilde{\bar{a}}(t)$ is defined as

$$\tilde{\bar{a}}(t) = \frac{\sum_i \bar{a}_i(t) N_i(t)}{\sum_i N_i(t)}. \tag{8.2}$$

The change over time of the average age of the entire population is due to two reasons. A change in the age structure "within" each subpopulation, and a change "between" the structure of the population given by the subpopulations. Which part of the change in the average age is due to changes "within" the subpopulations, and which part is due to changes "between" the subpopulations?

By applying equation (6.4) to (8.2) the decomposition of the change over time of the average age of the entire population is as follows

$$\dot{\tilde{\bar{a}}}(t) = \tilde{\dot{\bar{a}}} + C_N(\bar{a}, \bar{r}), \tag{8.3}$$

where the two terms on the right-hand side of the formula are weighted by the total subpopulation sizes $N_i(t)$. The first term of the decomposition $\tilde{\dot{\bar{a}}}$ is the average of the changes in the averages. This component can be further decomposed by applying equation (6.4) to each change in the average $\dot{\bar{a}} = C_n(a, r)$, presented in equation (6.2). The final decomposition is

$$\dot{\tilde{\bar{a}}}(t) = \widetilde{C_n(a, r)} + C_N(\bar{a}, \bar{r}). \tag{8.4}$$

The average of the covariance, $\widetilde{C_n(a, r)}$, is the direct effect and corresponds to the change "within" the subpopulations. This component explains the variation experienced in each of the subpopulations. The average of these covariances within each subpopulation is the level-1 effect. Note that the covariances are between ages and age-specific growth rates, $r_{ia}(t)$, and that the covariances use the subpopulation sizes $n_{ia}(t)$ as weighting functions.

The second covariance is the compositional component and corresponds to the change "between" the subpopulations. It captures the variations in the composition of the subpopulations. Note that the covariances are between average ages of the subpopulations, $\bar{a}_i(t)$, and the overall growth rates in the subpopulations, $\bar{r}_i(t)$, and they employ the total subpopulation sizes $N_i(t)$ as weighting functions.

Table 8.1 applies the decomposition in equation (8.4) to the change in the average age of the American population, divided into subpopulations. These subpopulations are defined by age and ethnic group: non-Hispanic white, non-Hispanic black, non-Hispanic Asian and Pacific Islander, American Indian and other non-Hispanic, and Hispanic. The table shows the

Table 8.1: Average age of United States population, $\tilde{\bar{a}}(t)$, calculated as the average over age and race, and decomposition of the annual change over time from 1990 to 2000.

$\tilde{\bar{a}}(1990)$	35.212
$\tilde{\bar{a}}(2000)$	36.491
$\dot{\tilde{\bar{a}}}$	0.128
$C_n\widetilde{(a,r)}$	0.162
$C_N(\bar{a},\bar{r})$	-0.034
$\dot{\tilde{\bar{a}}}(t) = C_n\widetilde{(a,r)} + C_N(\bar{a},\bar{r})$	0.128

Source: Author's calculations described in Chapter 9, based on the U.S. Census Bureau (2001). The population was divided in five subpopulations. These subpopulations are the different ethnic groups in the U.S.: non-Hispanic white, non-Hispanic black, non-Hispanic Asian and Pacific Islander, American Indian and other non-Hispanic, and Hispanic.

well-known aging process of the American population, evident in the positive values of the first term in equation (8.4), the average of the covariances. On average all the ethnic groups show increasing average age. This is the increase in the change "within" the ethnic groups.

On the other hand, the second term of (8.4) focuses on the different growth rates of the ethnic groups and their different average ages. The negative result of the covariance is due to the low average age and high population growth rate of Hispanics and the high average age and low growth rate of non-Hispanic whites. In other words, the lack of correspondence "between" the ethnic groups' average ages and growth rates results in a negative covariance.

This method of studying the heterogeneity "within" and "between" subpopulations can be simplified as the application of equation (6.4) two times. In the next section this extension is applied in the analysis of demographic variables that are not average ages.

8.2.2 Averages of Averages, Generalization

The previous section showed that an important case of averages is when they also involve averages. The development here is similar to the method proposed by Cho and Retherford (1973) to apply Kitagawa's decomposition formula twice. Here, we undertake a similar approach with equation (6.4).

Let x and z be two characteristics that divide the population into subpopulations. For example, in the previous section we had age and ethnic group. If $v(x,z,t)$ is a certain demographic function and $w(x,z,t)$ a weighting function, then the total average is the double integral over both characteristics x and z, expressed as

$$\tilde{\bar{v}}(t) = \frac{\int_0^\omega \bar{v}(x,t)W(x,t)dx}{\int_0^\omega W(x,t)dx}, \tag{8.5}$$

where $W(x,t)$ is the total weight over the characteristic z, $W(x,t) = \int_0^\omega w(x,z,t)dz$, and $\bar{v}(x,t)$ is the average over the characteristic z,

$$\bar{v}(x,t) = \frac{\int_0^\omega v(x,z,t)w(x,z,t)dz}{\int_0^\omega w(x,z,t)dz}. \tag{8.6}$$

It has to be noted that the order in which the indices are elected may not change the resultant overall average. This is proved by noting that $\tilde{\bar{v}}(t) = \bar{\tilde{v}}(t)$. Denoting $\int_x$ and $\int_z$ the integrals over x and z respectively and from the definition of the average of average $\tilde{\bar{v}}(t)$ we have

$$\tilde{\bar{v}}(t) = \frac{\int_0^\omega \bar{v}(x,t)W(x,t)dx}{\int_0^\omega W(x,t)dx} = \frac{\int_x\int_z v(x,z,t)w(x,z,t)dzdx}{\int_x\int_z w(x,z,t)dzdx}$$

and exchanging the integrals we obtain

$$= \frac{\int_z\int_x v(x,z,t)w(x,z,t)dxdz}{\int_z\int_x w(x,z,t)dxdz} = \frac{\int_0^\omega \tilde{v}(z,t)W(z,t)dz}{\int_0^\omega W(z,t)dz} = \bar{\tilde{v}}(t). \tag{8.7}$$

Nevertheless, the components of the decomposition may have different values depending on the selection of $\tilde{\bar{v}}(t)$ or $\bar{\tilde{v}}(t)$. The decomposition of the change over time of equation (8.5) is

$$\dot{\tilde{\bar{v}}}(t) = \tilde{\bar{\dot{v}}} + C_W(\bar{v},\bar{r}), \tag{8.8}$$

where $\bar{r}(x,t) = \acute{W}(x,t)$ is the intensity with respect to time of the weighting function $W(x,t)$, and $W(x,t)$ is the weight of both terms on the right-hand side, average and covariance. The term $\tilde{\bar{\dot{v}}}$ of equation (8.8) is the average of the key equation (6.4) of Chapter 6, $\dot{\bar{v}} = \bar{\dot{v}} + C(v,\acute{w})$. Therefore we can further express this result as

$$\dot{\tilde{\bar{v}}}(t) = \tilde{\bar{\dot{v}}} + \widetilde{C_w(v,r)} + C_W(\bar{v},\bar{r}), \tag{8.9}$$

where $r(x,z,t) = \acute{w}(x,z,t)$ is the intensity with respect to time of the weighting function $w(x,z,t)$. Another decomposition of the same average of averages is applied to the right side of equation (8.7). To find the decomposition of the change in the average $\bar{\tilde{v}}(t)$, when we take first the average over the values of x and then over z, we obtain

$$\dot{\bar{\tilde{v}}}(t) = \bar{\tilde{\dot{v}}} + \overline{C_w(v,r)} + C_W(\tilde{v},\tilde{r}). \tag{8.10}$$

Kim and Strobino (1984) noted the importance of hierarchy among the compositional components. If there is some factor which might be more important for the dynamics under study it is reasonable to try to capture first its contribution to the total change. Equations (8.9) and (8.10) allow the possibility of reflecting this hierarchy among the compositional components. Choosing among them depends on the relevance of the variables x or z in the study.

By applying equation (8.9) to the formula of the total average age over subpopulations (8.2) and taking into account that $\dot{a} = 0$, equation (8.4) is obtained. The change in the average age of the population can be expressed as an average of covariances plus a covariance of averages

$$\dot{\tilde{\bar{a}}}(t) = \widetilde{C_n(a, r)} + C_N(\bar{a}, \bar{r}). \tag{8.11}$$

Equations (8.9) and (8.10) show the two possibilities of the decomposition for the change over time of $\tilde{\bar{v}}$ and $\bar{\tilde{v}}$, respectively. The average age of the population can only be expressed as one of these averages $\tilde{\bar{a}}$. This is a consequence of the version of $\bar{\tilde{v}}$ applied to the average age of the population which is no longer an average of averages, but is only one average $\bar{\tilde{a}} = \bar{a}$. This follows because the averages over countries are simply equal to the fixed age,

$$\tilde{a}(t) = \frac{\sum_i a n_{ia}(t)}{\sum_i n_{ia}(t)} = a. \tag{8.12}$$

Therefore the decomposition of changes "within" and "between" the subpopulations in the average age of the population is uniquely defined by (8.11).

8.3 Simplifying a Complex Average

8.3.1 Crude Death Rate of a Group of Countries

In the applications of Chapter 6 two types of compositions of the population are separately studied. The composition component is in some examples the age structure of the population, and in other examples the composition of the population is by country of residence.

The previous section presented a method for studying both compositions of the population together. Here we present an alternative procedure. In this section it is shown that it is possible to separate the change over time in a demographic variable into two terms. One term corresponds to the direct effect and the other term accounts for compositional effects, age and country of residence all taken together.

Suppose the aim is to study the crude death rate of a group of countries, for example a selected group of European countries. The CDR of these European countries, denoted as $\bar{d}_E(t)$, can be calculated as an average of the crude death rates of the countries, $d_c(t)$, weighted by the population sizes of each country $N_c(t)$, with $c \in E$ indicating that all selected countries in E are considered, $c = 1, ..., 14$,

$$\bar{d}_E(t) = \frac{\sum_{c \in E} d_c(t) N_c(t)}{\sum_{c \in E} N_c(t)}. \tag{8.13}$$

On the other hand, the CDR for every country, $d_c(t)$, is also an average of the age-specific death rates over age, $m_{ac}(t)$. As in equation (2.7) for country c we have

$$d_c(t) = \frac{\sum_{a=0}^{\omega} m_{ac}(t) N_{ac}(t)}{\sum_{a=0}^{\omega} N_{ac}(t)}, \tag{8.14}$$

where a has 12 age groups 0 to 1, then 1-9, 10-19, and so on until 90-99 and 100 and above. The term $m_{ac}(t)$ is the age- and country-specific death rate and $N_{ac}(t)$ is the population size of country c and age a at time t. Substituting (8.14) in equation (8.13) changes an average of averages to an average. This is obtained by substituting the CDR of every country and keeping in mind that the total population size of each country is equal to the addition of the population sizes at all ages $N_c(t) = \sum_{a=0}^{\omega} N_{ac}(t)$,

$$\bar{d}_E(t) = \frac{\sum_{c \in E} \sum_{a=0}^{\omega} m_{ac}(t) N_{ac}(t)}{\sum_{c \in E} \sum_{a=0}^{\omega} N_{ac}(t)}. \tag{8.15}$$

For every age a and every country c we could define one level of a new variable k in the cross-classified contingency table of ages and countries. The total number of cells in this cross-classification comprises 148 levels, the product of 14 countries by 12 age groups, the product of c and a. The average of averages changes to an average of two single arguments, time t and the variable k that depends on age and country,

$$\bar{d}_E(t) = \frac{\sum_{k=1}^{ac} m_k(t) N_k(t)}{\sum_{k=1}^{ac} N_k(t)}. \tag{8.16}$$

This allows a simpler decomposition of the kind in equation (6.4),

$$\dot{\bar{d}}_E = \bar{\acute{m}} + C(m, r), \tag{8.17}$$

where $r(t) = r_k(t)$ is the age- and country-specific growth rate, $r_k(t) = \acute{N}_k(t)$.

Table 8.2 shows the change in the CDR of selected European countries and the decomposition of these changes. The countries are: Austria, Bulgaria, Finland, France, East and West Germany separately, Hungary, Italy, the Netherlands, Norway, Russia, Sweden and Switzerland. The analyzed periods are from 1960 to 1970 denoted in the table as 1965, from 1975 to 1985 denoted as 1980, and from 1992 to 1997 denoted as 1994. Russia was included in the last two columns of the table. The crude death rate of these selected European countries increased in all the studied periods. In the first two columns, 1965 and 1980, there are important changes in survivorship with negative average changes. On average all the countries' mortality rates decreased, seen in the $\bar{\acute{m}}$ term. But there were also significant contributions of the compositional component that opposed the decrease. As a result the CDR of these selected countries increased during the periods. For the last period, 1992 to 1996 in column 1994, both terms contribute to the increase in the crude death rate. In all the studied periods in Table 8.2 the covariance component is the main contributor to the observed change. These covariance accounts for the age structure and the population distribution over countries.

An immediate question the separation of the contribution of the age structure and the population structure over countries.

In the next section the extension of equation (6.4) that accounts for each of the compositional components is shown. Before carrying out further extension we show how it is possible to simplify a complex rate when more than two compositional components are involved.

Table 8.2: Crude death rate, $\bar{d}_E(t)$, per thousand, and decomposition of the annual change over time in 1960-1970, 1975-1985 and 1992-1996 for selected European countries.

	1965	1980	1994
$\bar{d}_E(t)$	10.783	10.720	11.187
$\bar{d}_E(t-h/2)$	10.720	10.591	10.916
$\bar{d}_E(t+h/2)$	10.892	10.964	11.594
$\dot{\bar{d}}_E(1994)$	0.017	0.037	0.170
$\dot{\bar{m}}$	-0.090	-0.112	0.036
$C(m,r)$	0.108	0.150	0.133
$\dot{\bar{d}}_E = \dot{\bar{m}} + C(m,r)$	0.018	0.038	0.169

Source: Author's calculations described in Chapter 9, based on the Human Mortality Database (2002). For the years 1965 and 1980 eleven-year periods were used (1960-1970 and 1975-1985). For the year 1994 a five-year period was used (1992-1996). The countries included are: Austria, Bulgaria, Finland, France, East and West Germany separately, Hungary, Italy, the Netherlands, Norway, Russia, Sweden and Switzerland. Russia was only included in the last two columns of the table.

8.3.2 Simplifying a Complex Average, Generalization

In general it is possible to change the additions over indexes x_1, x_2, ..., x_m for a single addition. Let x_1 have n_1 categories, x_2 have n_2 categories and so on until x_m with n_m categories. The new single addition over a new index k has a total number of categories equal to the product of $n_1 n_2 ... n_m$. Assuming that the demographic variable $\bar{v}(t)$ can be written as

$$\bar{v}(t) = \frac{\sum_{x_1}\sum_{x_2}...\sum_{x_m} v_{x_1 x_2 ... x_m}(t) w_{x_1 x_2 ... x_m}(t)}{\sum_{x_1}\sum_{x_2}...\sum_{x_m} w_{x_1 x_2 ... x_m}(t)},$$

where $\sum_{x_i}$ is the addition over the different levels of x_i. Define the new variable k as a function of the different levels of x_1, x_2,... and x_m combined. We obtain a simplified version of $\bar{v}(t)$ as

$$\bar{v}(t) = \frac{\sum_{k=1}^{n_1 n_2 ... n_m} v_k(t) w_k(t)}{\sum_{k=1}^{n_1 n_2 ... n_m} w_k(t)}, \tag{8.18}$$

where $\sum_{k=0}^{n_1 n_2 ... n_m}$ indicates that k has $n_1 n_2 ... n_m$ levels.

Equation (6.4) can now be applied without regard to the number of compositional components included. The new result has two components, one for the direct effect of the variable of interest and a second one for all the compositional components taken together.

In the study of the crude death rate it could be possible to include age structure, ethnic group structure, social class structure, education structure and country of residence. By applying equation (8.18) the composition changes to one variable k which includes all the structures: age, ethnic group, social class, education and country of residence.

This procedure of changing the complexity of the multitudinous dimensions of population heterogeneity to a simpler formulation allows a further decomposition of the compositional component. This further decomposition separates the compositional component to allow the distinction between the numerous structures of the population. The next sections show different possibilities for attaining the separation of the compositional component.

8.4 Decomposing the Compositional Component

In equation (6.4) the compositional effect, $C(v, \acute{w})$, accounts for the multitudinous dimensions of population heterogeneity combined. Here an elegant formula for separating the combined compositional components is shown.

Let the reader be reminded of the property of the relative derivative of a product

$$\frac{\frac{\partial}{\partial t}uv}{uv} = \acute{u} + \acute{v}, \tag{8.19}$$

and the covariance property of separating additions

$$C(u_1 + u_2, v) = C(u_1, v) + C(u_2, v), \tag{8.20}$$

which are two properties that help to achieve the decomposition of the compositional components.

Let us assume that there are n compositional components and let the weighting function $w(t)$ be a product of weighting functions $w = w_1 w_2 ... w_n$, where each weighting function $w_i(t)$ accounts for one of the compositional factors. Following the property of intensities (8.19) the intensity of the weighting function is replaced by the addition of the intensities $\acute{w} = \acute{w}_1 + \acute{w}_2 + ... + \acute{w}_n$. Substituting this result in our equation (6.4) and taking into account the property of covariances in (8.20) we have

$$\dot{\bar{v}} = \bar{\dot{v}} + C(v, \acute{w}_1 + \acute{w}_2 + ... + \acute{w}_n)$$

$$\dot{\bar{v}} = \bar{\dot{v}} + C(v, \acute{w}_1) + C(v, \acute{w}_2) + ... + C(v, \acute{w}_n). \tag{8.21}$$

Again here the first term on the right-hand side of (8.21), the average of the changes, captures the change in the characteristic of interest, this is the direct change. The second component is the covariance term between the underlying variable of interest and the intensity of the weighting function of the compositional factor w_1. This is the structural or compositional component of change due to the compositional factor w_1. In a similar way the third component is the compositional component of change due to compositional factor w_2, and so on until the compositional component of change due to compositional factor w_n.

This formulation is applied in the study of the CDR for the selected European countries. Let us assume that the weighting functions $N_k(t)$ in (8.17) can be separated into a product of age structure weights, $N_a(t)$, and country-structure weights $N_c(t)$, $N_k(t) = N_a(t)N_c(t)$. By applying (8.21) to equation (8.17) the CDR is further decomposed as

$$\dot{\bar{d}}_E = \bar{\dot{m}} + C(m, r_a) + C(m, r_c), \tag{8.22}$$

where the growth rates r_a and r_c correspond to the intensities of the age-structure weights $r_a(t) = \acute{N}_a(t)$, and country-structure weights $r_c(t) = \acute{N}_c(t)$, respectively.

The following sections show how to obtain a weighting function $w(a, t)$ as a product of weighting functions $w = w_1 w_2 ... w_n$.

8.5 Separating the Weighting Functions

8.5.1 Two Compositional Factors

Let us look at the case of two compositional factors x and z. The weights, $w_{xz}(t)$, are cross-classified by these two factors. To separate these weights into weights that account independently for the factors x and z, we can follow the technique suggested by Das Gupta (1994).

Let the marginal values of the two-way table be defined as $w_{x\cdot}(t) = \sum_z w_{xz}(t)$ and $w_{\cdot z}(t) = \sum_x w_{xz}(t)$. The trivial case is when each element of the cross-classification can be estimated as the product of the corresponding marginal values of the weighting function, in other words, when $w_{xz}(t) = w_{x\cdot}(t)w_{\cdot z}(t)$. That is when the compositional factors are independent. Because in most cases the elements of the two-way table differ from the product of the marginal values it is necessary to correct for these errors.

For each element $w_{xz}(t)$ the marginal value $w_{x\cdot}(t)$ is corrected by the contribution of the $w_{xz}(t)$ in the corresponding marginal value of $w_{\cdot z}(t)$. This is achieved by the product of the marginal in x and the ratio of $w_{xz}(t)$ over the marginal in z, as $\frac{w_{xz}(t)}{w_{\cdot z}(t)}w_{x\cdot}(t)$. At the same time each $w_{\cdot z}(t)$ is corrected by the contribution of the $w_{xz}(t)$ in the corresponding marginal value of $w_{x\cdot}(t)$. The resultant two elements of the product are

$$w_{xz}(t) = \left[\frac{w_{xz}(t)}{w_{\cdot z}(t)}w_{x\cdot}(t)\right]^{1/2}\left[\frac{w_{xz}(t)}{w_{x\cdot}(t)}w_{\cdot z}(t)\right]^{1/2} = w_x(t)w_z(t), \tag{8.23}$$

where $w_x(t)$ and $w_z(t)$ are the new weighting functions that account for x and z independently, which substitute the weights $w_{xz}(t)$. When each element of the weighting function is divided by the total values of the weighting function $\frac{w_{xz}(t)}{w_{\cdot\cdot}(t)}$, then the weights are normalized. These normalized weights correspond exactly to those suggested by Das Gupta (1994) and they are shown in equation (3.15).

We can now separate the different components of the change in the CDR for the selected European countries. As shown in (8.22) the decomposition of the change in the CDR of the European countries is

$$\dot{\bar{d}}_E = \bar{\dot{m}} + C(m, r_a) + C(m, r_c),$$

where the growth rates are calculated from the age and country population size terms obtained from the substitution of $N_{ac}(t)$ in (8.23)

$$N_{ac}(t) = \left[\frac{N_{ac}(t)}{N_{\cdot c}(t)}N_{a\cdot}(t)\right]^{1/2}\left[\frac{N_{ac}(t)}{N_{a\cdot}(t)}N_{\cdot c}(t)\right]^{1/2} = N_a(t)N_c(t),$$

as $r_a(t) = \acute{N}_a(t)$ and $r_c(t) = \acute{N}_c(t)$.

The terms on the right-hand side of (8.22) account for the change in the CDR due to change in the age-specific death rates, changes in the age structure of the countries, and change in the country composition, respectively.

The decomposition of the change in the CDR of the selected European countries from 1960 to 1970, from 1975 to 1985 and from 1992 to 1996 are shown in Table 8.3. The contribution of the two compositional factors is very different. The age structure has the biggest share of

Table 8.3: Crude death rate, $\bar{d}_E(t)$, per thousand, and decomposition of the annual change over time in 1960-1970, 1975-1985 and 1992-1996 for selected European countries. Decomposing by direct change and two compositional components: age and country.

	1965	1980	1994
$\bar{d}_E(t)$	10.783	10.720	11.187
$\bar{d}_E(t-h/2)$	10.720	10.591	10.916
$\bar{d}_E(t+h/2)$	10.892	10.965	11.595
$\dot{\bar{d}}_E(t)$	0.017	0.037	0.170
$\bar{\dot{m}}$	-0.090	-0.112	0.036
$C(m,r)$	0.108	0.150	0.133
$C(m,r_a)$	0.110	0.148	0.134
$C(m,r_c)$	-0.002	0.002	-0.001
$\dot{\bar{d}}_E = \bar{\dot{m}} + C(m,r_a) + C(m,r_c)$	0.018	0.038	0.169

Source: Author's calculations described in Chapter 9, based on the Human Mortality Database (2002). For the years 1965 and 1980 eleven-year periods were used (1960-1970 and 1975-1985). For the year 1994 a five-year period was used (1992-1996). The countries included are: Austria, Bulgaria, Finland, France, East and West Germany separately, Hungary, Italy, the Netherlands, Norway, Russia, Sweden and Switzerland. Russia was only included in the last two columns of the table.

the compositional component. In 1965 and 1994 the age structure contributes to the increase of the CDR, while the country composition contributes in a minor way to the decrease of the CDR. The addition of these two terms gives the total compositional change of 0.108, 0.150 and 0.133 per thousand seen in Table 8.2.

Equation (8.21) can be used in many other applications. The following section shows the decomposition of average age at death by cause of death.

8.5.2 Decomposing the Average Age at Death

Valkovics (1999) studied the decomposition of the change in life expectancy at birth expressed as the mean age at death,

$$e^o(0,t) = \frac{\int_0^\omega \ell(a,t)da}{\ell(0,t)} = \frac{\int_0^\omega ad(a,t)da}{\int_0^\omega d(a,t)da}, \tag{8.24}$$

where $d(a,t)$ denotes the number of deaths at age a and time t in the lifetable, and $\ell(a,t)$ the survival function at age a and time t. equation (8.24) shows that life expectancy is also an average. Another expression of life expectancy as an average is presented by Vaupel and Canudas Romo (2003).

A similar demographic measure is the average age at death in the population. The number of deaths at any age is equal to the addition of the deaths due to different causes of death. In mathematical notation this is $D_{a\cdot}(t) = \sum_i D_{ai}(t)$ with i the cause of death. A new average can be obtained by exchanging sums for integrals and by inserting the sum over cause of death in equation (8.24). An alternative long averaging proposed in this chapter for the average age at death is

$$\bar{a}_D(t) = \frac{\sum_{a=0}^{\omega} \sum_i aD_{ai}(t)}{\sum_{a=0}^{\omega} \sum_i D_{ai}(t)} = \frac{\sum_{k=1}^{ai} aD_k(t)}{\sum_{k=1}^{ai} D_k(t)}, \tag{8.25}$$

where k depends on the age a and the causes of death i. Applying our main equation (6.4) the change over time in the average age at death is equal to the covariance between age and the intensity of the number of deaths

$$\dot{\bar{a}}_D(t) = C(a, \acute{D}). \tag{8.26}$$

From equation (8.21) the covariance term can be further decomposed into the change due to the distribution of deaths over age and another term for the distribution among the different causes of death

$$\dot{\bar{a}}_D(t) = C(a, \acute{D}_a) + C(a, \acute{D}_i), \tag{8.27}$$

where $\acute{D}_a$ and $\acute{D}_i$ are the intensities of the age structure of deaths and cause of death distribution respectively.

Take as an example the case of Japan. The proportion of deaths due to the different causes of death in the years 1980 and 1990 are shown in Table 7.4. Table 8.4 shows the average age at death in Japan and the decomposition obtained in (8.27) by distribution of deaths and causes of death, between 1980 and 1990. The change over time in the average age at death

Table 8.4: Average age at death, $\bar{a}_D(t)$, and decomposition of the annual change over time for Japan from 1980 to 1990. Decomposing by distribution of deaths over age and by cause of death.

$\bar{a}_D(1985)$	70.492
$\bar{a}_D(1980)$	68.629
$\bar{a}_D(1990)$	72.389
$\dot{\bar{a}}_D(1985)$	0.376
$C(a, \acute{D}_a)$	0.395
$C(a, \acute{D}_i)$	-0.019
$\bar{a}_D(t) = C(a, \acute{D}_a) + C(a, \acute{D}_i)$	0.376

Source: Author's calculations described in Chapter 9, based on the Berkeley Mortality Database (2001).

is mainly due to the first covariance $C(a, \acute{D}_a)$. This term corresponds to the distribution of deaths over age. The positive result in this term is due to a shift of deaths to older ages. The second covariance is between ages and the intensity of the cause of death distribution $C(a, \acute{D}_i)$. This term captures the change in the distribution of cause of death over age. The greatest changes in the distribution of cause of death between the studied years occur at the oldest age groups. By reducing the contribution of some causes of death the contribution of other causes increase. As a consequence for some causes of death the covariance with age is positive while for others is negative. In this application the negative covariance $C(a, \acute{D}_i)$ plays a minor role in the change of the average age at death.

8.5.3 Three Compositional Factors or More

In the case of three compositional factors it could be possible to do a similar general case using Das Gupta's formulations. The case of three compositional factors x, y and z, also follows equation (8.23) by having first the estimation of the elements in the three-way table as the product of marginal values

$$w_{xyz}(t) = w_{x\cdot\cdot}(t)w_{\cdot y\cdot}(t)w_{\cdot\cdot z}(t). \tag{8.28}$$

Following Das Gupta's methodology the marginal value $w_{x\cdot\cdot}(t) = \sum_{y,z} w_{xyz}(t)$ is corrected for the undesired interactions. The cell of the data $w_{xyz}(t)$ is corrected by the interaction marginal value of the other two components $w_{\cdot yz}(t) = \sum_x w_{xyz}(t)$, as $\frac{w_{xyz}(t)}{w_{\cdot yz}(t)}$. But also the interactions including x have to be corrected for other undesired components, $\frac{w_{xy\cdot}(t)}{w_{\cdot y\cdot}(t)}$ and $\frac{w_{x\cdot z}(t)}{w_{\cdot\cdot z}(t)}$. The same applies to the other marginal values $w_{\cdot y\cdot}(t) = \sum_{x,z} w_{xyz}(t)$ and $w_{\cdot\cdot z}(t) = \sum_{x,y} w_{xyz}(t)$. The three new weights are

$$\begin{aligned}
w_{xyz}(t) &= \left[\frac{w_{xyz}(t)}{w_{\cdot yz}(t)}\left[\frac{w_{xy\cdot}(t)}{w_{\cdot y\cdot}(t)}\frac{w_{x\cdot z}(t)}{w_{\cdot\cdot z}(t)}\right]^{1/2} w_{x\cdot\cdot}(t)\right]^{1/3} \\
&\quad \left[\frac{w_{xyz}(t)}{w_{x\cdot z}(t)}\left[\frac{w_{xy\cdot}(t)}{w_{x\cdot\cdot}(t)}\frac{w_{\cdot yz}(t)}{w_{\cdot\cdot z}(t)}\right]^{1/2} w_{\cdot y\cdot}(t)\right]^{1/3} \\
&\quad \left[\frac{w_{xyz}(t)}{w_{xy\cdot}(t)}\left[\frac{w_{x\cdot z}(t)}{w_{x\cdot\cdot}(t)}\frac{w_{\cdot yz}(t)}{w_{\cdot y\cdot}(t)}\right]^{1/2} w_{\cdot\cdot z}(t)\right]^{1/3} \\
&= w_x(t)w_y(t)w_z(t).
\end{aligned} \tag{8.29}$$

Table 8.5 shows the CDR for the selected European countries and the decomposition of the change over time. The three compositional factors included are age, country and sex and the corresponding decomposition formula is

$$\dot{\bar{d}}_E = \bar{\dot{m}} + C(m, r_a) + C(m, r_c) + C(m, r_s), \tag{8.30}$$

where r_a, r_c and r_s correspond to the growth rates of the population accounting only for the compositional effect of the countries, ages and sexes respectively. As in Table 8.3 the main compositional component is the age structure, with the country and sex composition almost canceling each other. It has to be noted that in Tables 8.2, 8.3 and 8.5 the death rates and the population size are over three categories, namely, age, country and sex, $m_{acs}(t)$ and $N_{acs}(t)$. In Tables 8.2 and 8.3 only two of these compositional components are considered while in Table 8.5 all are involved. Therefore, the observed average change and compositional component

Table 8.5: Crude death rate, $\bar{d}_E(t)$, per thousand, and decomposition of the annual change over time in 1960-1970, 1975-1985 and 1992-1996 for selected European countries. Decomposing by direct change and three compositional components: age, sex and country.

	1965	1980	1994
$\bar{d}_E(t)$	10.783	10.720	11.187
$\bar{d}_E(t-h/2)$	10.720	10.591	10.916
$\bar{d}_E(t+h/2)$	10.892	10.965	11.595
$\dot{\bar{d}}_E(t)$	0.017	0.037	0.170
$\bar{\dot{m}}$	-0.090	-0.112	0.036
$C(m,r)$	0.108	0.150	0.133
$C(m,r_a)$	0.110	0.148	0.134
$C(m,r_c)$	-0.001	0.001	-0.006
$C(m,r_s)$	-0.001	0.001	0.005
$\dot{\bar{d}}_E = \bar{\dot{m}} + C(m,r_a) + C(m,r_c) + C(m,r_s)$	0.018	0.038	0.169

Source: Author's calculations described in Chapter 9, based on the Human Mortality Database (2002). For the years 1965 and 1980 eleven-year periods were used (1960-1970 and 1975-1985). For the year 1994 a five-year period was used (1992-1996). The countries included are: Austria, Bulgaria, Finland, France, East and West Germany separately, Hungary, Italy, the Netherlands, Norway, Russia, Sweden and Switzerland. Russia was only included in the last two columns of the table.

are the same in the three tables. Different results are obtained if the death rates and the population sizes were defined only for two characteristics first and then for three.

Another application of the generalization of the decomposition method is shown in Table 8.6 for the general fertility rate for selected European countries, $GFR_E(t) = \bar{b}_E(t)$. The births of Denmark, France, the Netherlands and Sweden are divided over the total population of females in reproductive ages α and β. The fertility rates are divided by age, marital status (married and unmarried), and by country. Therefore, as done for the CDR of selected European countries now we can decompose the $GFR_E(t)$ into a direct effect of change in the fertility rates and three compositional components. First the $GFR_E(t)$ is simplified into a rate with only one compositional component k which is a function of age, marital status and country,

$$\bar{b}_E(t) = \frac{\sum_{a=\alpha}^{\beta} \sum_s \sum_c B_{asc}(t)}{\sum_{a=\alpha}^{\beta} \sum_s \sum_c N_{fasc}(t)} = \frac{\sum_{k=1}^{asc} b_k(t) N_{fk}(t)}{\sum_{k=1}^{asc} N_{fk}(t)} \tag{8.31}$$

where $N_{fasc}(t)$ and $B_{asc}(t)$ are the female population size and the births from mothers at age a, marital status s and country c. $N_{fk}(t)$ and $b_k(t)$ are the female population size and fertility rates of the compositional component k.

Then the decomposition in equation (6.4) is used to obtain a direct and a compositional component

$$\dot{\bar{b}}_E(t) = \bar{\dot{b}}(t) + C(b,r), \tag{8.32}$$

where $r(t) = r_{fk}(t)$ is the age, marital status and country-specific growth rate, $r_{fk}(t) = \acute{N}_{fk}(t)$.

The compositional component is further decomposed into an age, marital status and country component as shown in this section,

$$\dot{\bar{b}}_E(t) = \bar{\dot{b}}(t) + C(b, r_a) + C(b, r_s) + C(b, r_c), \tag{8.33}$$

where r_a, r_s and r_c correspond to the growth rates of the population accounting only for the compositional effect of the ages, marital status and countries respectively. Table 8.6 presents the decomposition of the general fertility rate for Denmark, France, the Netherlands and Sweden taken together when different compositional components are considered. Similar to Table 8.2 the compositional component explains most of the change in the general fertility rate in Table 8.6. Once this compositional component is separated we obtain half of this component due to change in the marital status composition and the other half due to age composition.

Table 8.6: General fertility rate, $\bar{b}_E(t)$, per hundredth, and decomposition of the annual change over time in 1992-1997 for selected European countries. Decomposing by direct change and three compositional components: age, marital status and country.

Compositional component	1	2	3
$\bar{b}_E(1994)$	5.042	5.042	5.042
$\bar{b}_E(1992)$	5.212	5.212	5.212
$\bar{b}_E(1997)$	4.925	4.925	4.925
$\dot{\bar{b}}_E(t)$	-0.029	-0.029	-0.029
$\bar{\dot{b}}(t)$	0.029	0.029	0.029
$C(b, r)$	-0.057	-0.057	-0.057
$C(b, r_a)$		-0.027	-0.027
$C(b, r_s)$		-0.030	-0.031
$C(b, r_c)$			0.001
$\dot{\bar{b}}_E(t) = \bar{\dot{b}}(t) + C(b, r_a) + C(b, r_s) + C(b, r_c)$	-0.028	-0.028	-0.028

Source: Author's calculations described in Chapter 9, based on Eurostat (2000).

8.5.4 Purging the Weighting Functions

Separating the weighting function into multiplicative components can be done in several ways. The previous section showed how the implementation of Das Gupta's formulations yields one of these solutions. Here, to separate the weights into terms that account independently for each of the compositional components we follow the use of a relationship model. This tool is the basis of the technique of "purging" undesirable compositional effect suggested by Clogg (1978).

In Chapter 5 it is shown that expected frequencies of a contingency table can be substituted by parameters of a general multiplicative model. Let a two-way contingency table be classified by two compositional variables. Let F_{ij} denote the expected frequencies in the cells (i, j) in the two-way contingency table. The general log-linear model in this two-way contingency table is

$$F_{ij} = \eta \tau_i \tau_j \tau_{ij}, \tag{8.34}$$

where η is the scale factor and the τ parameters denote various kinds of main effects and interaction. The parameter τ_{ij} is introduced here to account for the specific information of each cell which is not captured by the main effects.

The tabulated data of the weighting functions $w_{xz}(t)$ classified by x and z are elements of the two-way table. The log-linear model in equation (8.34) applied to the weighting functions gives

$$w_{xz} = \eta \tau_x \tau_z \tau_{xz}, \tag{8.35}$$

where the parameters τ_x and τ_z are new weighting functions that account for x and z independently, and τ_{xz} is an interaction parameter.

The final decomposition is reached by following the property of intensities (8.19), $\acute{w}_{xz} = \acute{\eta} + \acute{\tau}_x + \acute{\tau}_z + \acute{\tau}_{xz}$, and (8.21) to obtain,

$$\dot{\bar{v}} = \bar{\dot{v}} + C(v, \acute{\eta}) + C(v, \acute{\tau}_x) + C(v, \acute{\tau}_z) + C(v, \acute{\tau}_{xz}). \tag{8.36}$$

Nevertheless, this formulation presents the inconvenience of rapidly increasing the number of compositional parameters when the number of compositional variables increases. For the case of three compositional variables the number of compositional parameters in the saturated model is eight. In general, with n compositional components there are 2^n different parameters. It could be possible to use non-saturated models and thus obtain fewer parameters but the expenses of greater error. Das Gupta (1993) suggests that the interaction terms are negligible and therefore, equation (8.36) simplifies only to covariances with main effects.

For example, by applying equation (8.36) to the change over time in the CDR of the selected European countries five terms are obtained

$$\dot{\bar{d}}_E = \bar{\dot{m}} + C(m, \acute{\eta}) + C(m, \acute{\tau}_a) + C(m, \acute{\tau}_c) + C(m, \acute{\tau}_{ac}). \tag{8.37}$$

In equation (8.37) the first term on the right-hand side accounts for the change in the CDR due to change in the age-specific death rates, while the covariances correspond to the compositional components.

A visual representation of a multidimensional decomposition is shown in the next section.

8.6 Categorical Decomposition in a Two-Way Table

The reason for the high compositional effect in Table 8.2 can be seen through simultaneous analysis of an age and category decomposition. In Chapter 7 age, single age and categorical decompositions were presented. Here we extend this idea to the case of numerous compositional components.

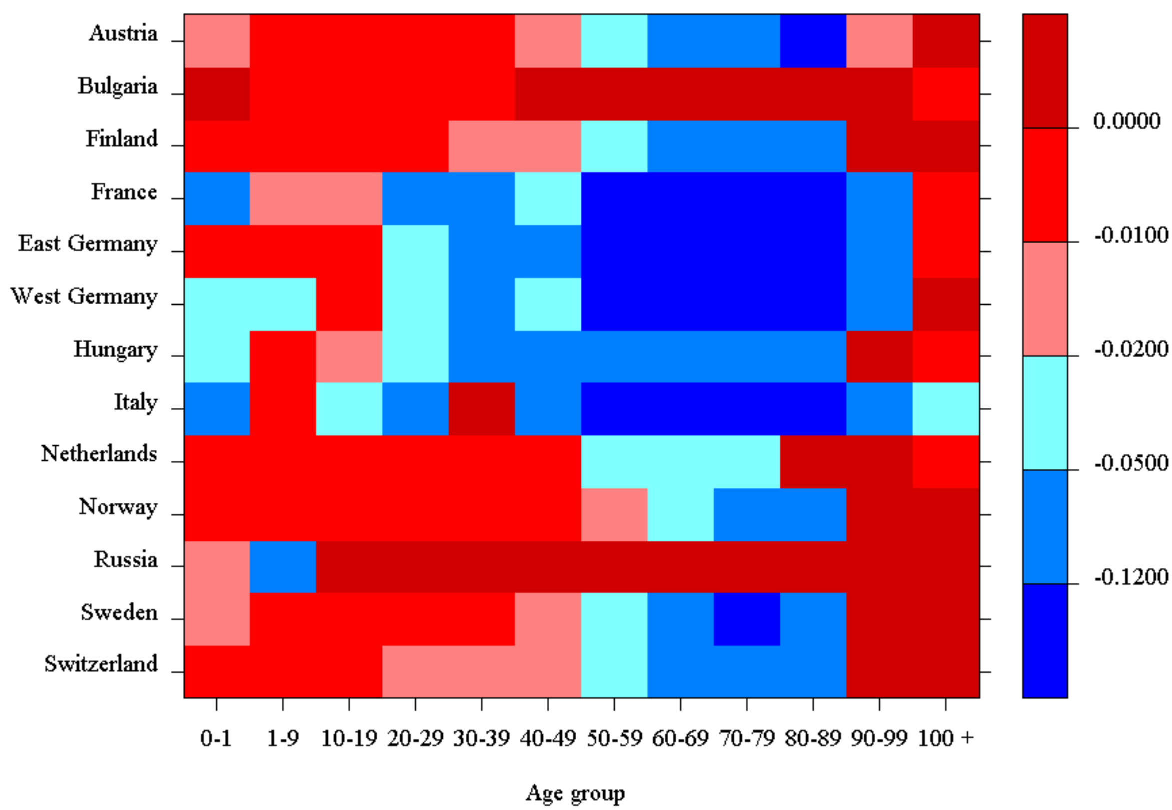
The same formulas used in Chapter 7 for decomposing the direct and compositional component are employed to compile the results of Table 8.2. In this way it is possible to know the contribution of each country and age group to the total change of the CDR. Also it is possible to obtain the contribution of the age and country category in the direct component and compositional component.

The graphic representation of the age and country decomposition is a surface with age groups in the horizontal axis, countries in the vertical axis and levels of these crossings as the surface points. The use of shaded contour maps is implicit in some of Lexis's (1875) original diagrams. Arthur and Vaupel (1984) introduced the phrase "Lexis surface" to describe a surface of demographic rates defined over age and time and we continue to employ that usage here.

The lexis surface is a useful tool that allows a flat representation of surfaces and therefore is perfect for the purpose of showing decomposition over two compositional factors.

Figures 8.1 and 8.2 show the age and country decompositions of the level-1 and level-2 components of change in the CDR, respectively. To visualize these contributions and because some values were extremely low, we show the results multiplied per 100,000. In Figure 8.1 blue

Figure 8.1: Lexis surface of an age and country decomposition of the direct component of the annual change in the crude death rate of selected European countries from 1992 to 1996.

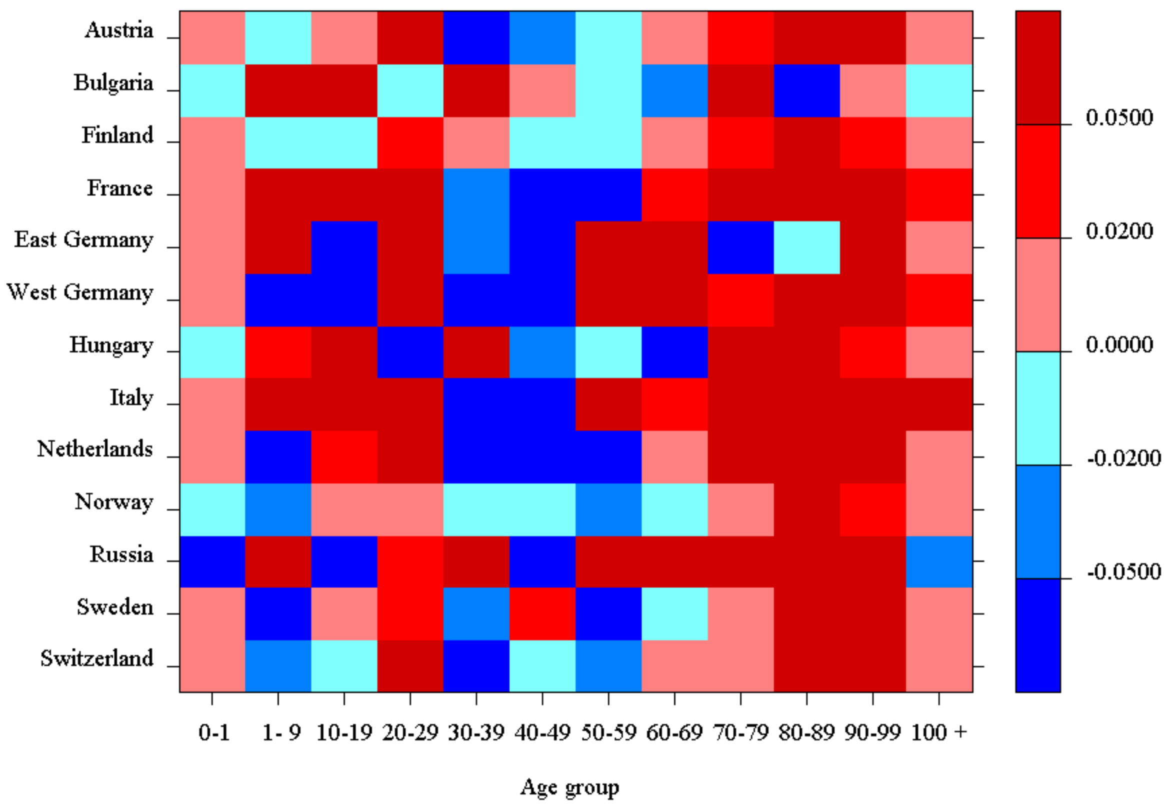


corresponds with negative numbers while red with numbers associated with positive values. It is possible to discern that these European countries have not contributed in the same way to the change in the total CDR. France, Italy, East and West Germany and Hungary are the greatest contributors to the average decline in mortality rates. On the other hand, Bulgaria, Russia and the Netherlands show fewer improvements in survivorship, with a greater number of red boxes.

In Figure 8.1 the improvements previously mentioned are concentrated in the age groups 40 to 99 while young age groups and very old age groups experienced the opposite trend, a decrease in mortality.

In a similar way the compositional component can be decomposed by country and age as shown in Figure 8.2. The scale has changed, but the colors still follow the same pattern. Blue

Figure 8.2: Lexis surface of an age and country decomposition of the compositional component of the annual change in the crude death rate of selected European countries from 1992 to 1996.



indicates a contribution to the decline in the total change in the CDR and red indicates an increase. It is then evident that those age groups that have a compositional component that opposes the decrease in the CDR are those aged 60 and above. The age groups 30 to 60 contribute to the decrease in the total CDR. For the youngest age groups 0 to 1 there are no changes in the compositional components.

8.7 Conclusion

Populations differ by age, sex, race, country of residence, religion, education, and countless other characteristics. Populations often undergo important compositional changes over time that affect comparisons of demographic variables over time. A formula for decomposing changes in a population average into different components is presented. One component captures the effect of direct change in the characteristic of interest, and the others capture the effect of the different compositional effects.

To capture the contribution of a single category to the total change equation (6.4) is further decomposed by age groups and categories. Applications of the newly presented generalization included the decomposition of the change over time in the average age of the United States population, the average age at death in Japan, as well as, the GFR and the CDR for selected countries in Europe. For the last example a categorical decomposition by countries and age groups is also included.

The beginning of this chapter introduced a straightforward solution for the change in averages that contain averages. The solution is to apply equation (6.4) two times. Another solution for the case of combining multitudinous dimensions of population heterogeneity is presented in the rest of the chapter. The proposed solution is an elegant separation of the weighting function into multiplicative components; however, this separation can not be done uniquely. The final two sections present two different ways of separating the weighting function to account independently for the compositional factors.