Part III

Decomposing Demographic Change Into Direct Versus Compositional Components

6

Decomposing Demographic Averages

6.1 Introduction

This chapter presents direct vs. compositional decomposition developed by Vaupel (1992) and further extensions by this author. This decomposition has been applied to data by Vaupel and Canudas Romo (2002) and the present chapter as well as following chapters include this author's contributions to the field by extending Vaupel's formulation. As already noted we called this method "the direct versus compositional decomposition" because it separates the change over time into these components.

First we introduced the sources that inspired the development of the decomposition. Vaupel's main formula and the interpretation of its components are presented in Section 6.2. This main formula can be extended to any kind of compositional structure, not only to the age structure, as illustrated, by using averages over subpopulations in Section 6.3. In Section 6.4 we examine the derivatives of demographic functions, such as differences and additions. A formula for relative changes is then introduced in Section 6.5 and the study of relative changes of functions is explained in Section 6.6. Finally a new decomposition of life expectancy is displayed in Section 6.7.

In each section where formulas are presented, applications are also included. In the last part of the book, a comparison is made between direct vs. compositional decomposition and the previous methods shown in Part II. To ensure uniformity and comparability, the applications seen here in Part III use the same data as the applications of Part II which is entitled *Decomposition Methods*.

To study population aging, Preston, Himes and Eggers (1989) analyzed the change over time in the average age of the population. The average age of the population is

$$\bar{a}(t) = \frac{\int_0^\omega aN(a,t)da}{\int_0^\omega N(a,t)da}, \tag{6.1}$$

where as before $N(a,t)$ is the population size at age a and time t.

By differentiating this expression with respect to time they found this change equal to the covariance between ages and age-specific growth rates $r(a,t)$,

$$\dot{\bar{a}} = C(a,r), \tag{6.2}$$

where the covariance is as defined in (2.12). The average age of the population increases for aging populations. This can also be seen in equation (6.2) where $\dot{\bar{a}}$ increases when the covariance is positive, that is, when the age-specific growth rates increase with age.

The findings of Preston, Himes and Eggers were extended by Vaupel (1992), and applied by Vaupel and Canudas Romo (2002). In the following section the proof of this generalization is presented together with some illustrations.

6.2 Derivatives of Averages

The object of interest is the change over time of demographic variables. Let $\bar{v}(t)$ be a demographic average as defined in equation (2.1). We analyze the change in an average by studying its derivative with respect to time t. The change in the average, $\dot{\bar{v}}$, is

$$\dot{\bar{v}} = \frac{\partial}{\partial t} \frac{\int_0^\infty v(x,t)w(x,t)dx}{\int_0^\infty w(x,t)dx}. \tag{6.3}$$

The decomposition of the change over time of a demographic variable can be simply and memorably expressed as the sum of two components: the average change plus the covariance between the variable of interest and the intensity of the weighting function,

$$\dot{\bar{v}} = \bar{\dot{v}} + C(v, \acute{w}). \tag{6.4}$$

The average change, $\bar{\dot{v}}$, is

$$\bar{\dot{v}} = \frac{\int_0^\infty \left[\frac{\partial}{\partial t} v(x,t)\right] w(x,t)dx}{\int_0^\infty w(x,t)dx}, \tag{6.5}$$

and the covariance is calculated as

$$C(v,\acute{w}) = \frac{\int_0^\infty \left(\left[v(x,t) - \bar{v}(t)\right]\left[\acute{w}(x,t) - \bar{\acute{w}}(t)\right]\right) w(x,t)dx}{\int_0^\infty w(x,t)dx}$$

$$= \frac{\int_0^\infty v(x,t)\acute{w}(x,t)w(x,t)dx}{\int_0^\infty w(x,t)dx} - \frac{\int_0^\infty v(x,t)w(x,t)dx}{\int_0^\infty w(x,t)dx}\frac{\int_0^\infty \acute{w}(x,t)w(x,t)dx}{\int_0^\infty w(x,t)dx}. \tag{6.6}$$

The derivation of equation (6.4) follows from substitution of (6.3), (6.5) and (6.6) into the corresponding terms in (6.4). Another proof of the derivation of equation (6.4) can be seen in Vaupel and Canudas Romo (2002).

Equation (6.4) tells us that the change in the average, $\dot{\bar{v}}$, can be decomposed into two terms. The first term, $\bar{\dot{v}}$, the average change, accounts for the change observed in the population produced by a direct change in the characteristic of interest. In the rest of the text we refer to this term as direct change or the level-1 effect of change.

The second component, $C(v, \acute{w})$, the covariance term, is the structural or compositional component of change; it accounts for the changes in population heterogeneity. In the remaining text this compositional term is called compositional component or the level-2 effect of change. Schoen and Kim (1992) also derived a formula, similar to (6.4), in which there is only compositional change and no direct change.

The crude death rate, $\bar{\mu}(t)$ also denoted $d(t)$, defined in equation (2.7) is a demographic average (see Chapter 2 for this discussion). It follows directly from equation (6.4) that the change over time of the variable, $\dot{\bar{\mu}}$, is decomposed into two terms. These are the average changes in the mortality rates, $\bar{\dot{\mu}}$, plus the covariance between mortality rates and growth rates,

$$\dot{\bar{\mu}} = \bar{\dot{\mu}} + C(\mu, r), \tag{6.7}$$

where $r(a,t)$ are the age-specific growth rates $r(a,t) = \acute{N}(a,t)$.

Table 6.1 presents an application of equation (6.7) and it shows decomposition of the change in the crude death rate for the member countries of the North American Free Trade Agreement (NAFTA) from 1985 to 1995.

Equation (6.7) is in continuous form but demographic data are discrete, so we estimated the values in the table using the methods described in Chapter 9. The estimated values are sometimes slightly different from the observed figures. These discrepancies arise when discrete data over a h-year period are used to approximate derivatives and averages at a particular instant. In all the tables of this part of the book there is a line dividing observed and estimated values. Because the change occurs continuously from the initial moment of 1985 to the final moment of 1995, we choose to study the change at the mid-year, here 1990. The values of the average at all these years are displayed together with the observed changes. The lower part of the table corresponds to the estimated data. First, the estimated components are presented followed by the estimated change.

The three countries that are members of the NAFTA experienced a decrease in crude death rates between 1985 and 1995. The United States underwent the highest decrease followed by Mexico and Canada in third position with a minor change. These changes are mainly due to a decrease in the age-specific death rates, or level-1 change. In Canada and Mexico the average change in the mortality rates, $\bar{\dot{\mu}}$, is about the same while the United States experiences a very marked improvement.

The change in the age structure of the three populations is contrary to the level-1 changes. For Canada the level-2 effect is almost as high as the average change in mortality rates and when added to the level-1 effect, to estimate the total change, results in a very small change in the Canadian $\dot{\bar{\mu}}$. Canada's reductions in mortality were balanced out by the aging process of its population.

Table 6.1: Crude death rate, $d(t)$, per thousand, and decomposition of the annual change over time from 1985 to 1995 for Canada, Mexico and United States.

	Canada	Mexico	United States
$d(1990)$	7.150	5.100	8.762
$d(1985)$	7.195	5.532	9.683
$d(1995)$	7.116	4.755	7.941
$\dot{d}(1990)$	-0.008	-0.078	-0.174
$\bar{\dot{\mu}}$	-0.109	-0.116	-0.245
$C(\mu, r)$	0.101	0.039	0.071
$\dot{d} = \bar{\dot{\mu}} + C(\mu, r)$	-0.008	-0.077	-0.174

Source: Author's calculations described in Chapter 9, based on the United Nations Data Base (2001).

In many situations the demographic average in equation (2.1) is described as the product of two terms

$$\bar{v}(t) = \int_0^\omega v(x,t)c(x,t)dx, \tag{6.8}$$

where $c(x,t)$ denotes the proportion of the total values of the weights that belong to the category x at time t, $c(x,t) = \frac{w(x,t)}{\int_0^\infty w(x,t)dx}$, and therefore $\int_0^\omega c(x,t)dx = 1$. Under these circumstances equation (6.4) changes to

$$\dot{\bar{v}} = \bar{\dot{v}} + \overline{v\acute{c}}. \tag{6.9}$$

This is easily proved by looking at the derivative of $\bar{v}(t)$, which follows the rule of the derivative of a product

$$\dot{\bar{v}} = \int_0^\omega \dot{v}(x,t)c(x,t)dx + \int_0^\omega v(x,t)\dot{c}(x,t)dx = \bar{\dot{v}} + \overline{v\left(\frac{\dot{c}}{c}\right)}. \tag{6.10}$$

As a consequence of equations (6.4) and (6.10) the covariance component $C(v, \acute{w})$ for the level-2 effect of change can also be expressed as

$$C(v, \acute{w}) = \overline{v\acute{c}}. \tag{6.11}$$

6.3 Averages over Subpopulations

The focus so far has been on averages over age. However age heterogeneity is only one of the multitudinous dimensions of population heterogeneity. In this section we present averages over another characteristic, namely population size by country of residence.

Consider a population composed of different subpopulations. Life expectancy at birth at time t for the entire population, $\bar{e}^o(t)$, is defined as the average of life expectancies for the subpopulations

$$\bar{e}^o(t) = \frac{\sum_i e_i^o(t) N_i(t)}{\sum_i N_i(t)}, \tag{6.12}$$

where $N_i(t)$ is the size of the subpopulation i and $e_i^o(t)$ is the subpopulation life expectancy at birth. Here, we use the subpopulation size, $N_i(t)$, as the weights.

The change in $\bar{e}_o$ over time can be decomposed following equation (6.4) as

$$\dot{\bar{e}}^o = \bar{\dot{e}}^o + C(e^o, r), \tag{6.13}$$

where $r_i(t)$ is the population growth rate of the ith subpopulation, $r_i(t) \equiv \acute{N}_i(t)$.

In Table 6.2 equation (6.13) is applied to changes in life expectancy of the chosen European countries. The common life expectancy at birth at time t for specific European countries is calculated as shown in equation (6.12). In Chapter 3 Tables 3.2, 3.3, 3.4 and 3.5 presented the crude death rate of specific European countries. In Table 6.2 we use the same countries to study the life expectancy for the entire population of these countries. The periods of observation are from 1960 to 1970, 1975 to 1985 and from 1992 to 1996, denoted by their mid-years 1965, 1980 and 1994. For the years 1965 and 1980, the common life expectancy

Table 6.2: Life expectancy at birth, $\bar{e}_o(t)$, and decomposition of the annual change over time in 1960-1970, 1975-1985 and 1992-1996. Life expectancy is calculated as the average over selected European countries.

t	1965	1980	1994
$\bar{e}^o(t)$	70.718	71.747	73.359
$\bar{e}^o(t-5)$	69.856	70.974	73.322
$\bar{e}^o(t+5)$	71.570	72.518	73.201
$\dot{\bar{e}}^o(t)$	0.171	0.154	-0.030
$\bar{\dot{e}}^o$	0.165	0.156	-0.043
$C(e^o, r)$	0.006	-0.002	0.013
$\dot{\bar{e}}^o = \bar{\dot{e}}^o + C(e^o, r)$	0.171	0.154	-0.030

Source: Author's calculations described in Chapter 9, based on the Human Mortality Database (2002). For the years 1965 and 1980 ten-year periods were used (1960-1970 and 1975-1985). For the year 1994 a four-year period was used (1992-1996). The countries included are: Austria, Bulgaria, Finland, France, East and West Germany separately, Hungary, Italy, the Netherlands, Norway, Russia, Sweden and Switzerland. Russia was only included in the last two columns of the table.

gained an annual increase of 17% and 15%, respectively. On the other hand, in 1994 we find a decline in life expectancy of 3%. The level-1 effect accounted for most of these changes, i.e.,whether there is an increase or decrease. In 1994, the level-2 effect shows a polarization of the subpopulations' life expectancies and growth rates. Countries which experience the highest population growth rates are also those with the longest life expectancy. The opposite is also true for those countries which have the lowest levels of growth rates and life expectancy. As a result of these relations between life expectancies and growth rates, the covariance term is positive.

In Chapter 4 some methods for decomposing life expectancy were shown. An alternative decomposition of the change in life expectancy analogous to equation (6.4) is shown by Vaupel and Canudas Romo (2003). This decomposition is explained in the last section of this chapter.

6.4 Derivatives of Demographic Differences and Additions

Equation (6.4) yields a simple but powerful result that divides the change over time of a demographic variable into points of main interest for demographers. The use of this general result is not restricted to averages but it could also be used on functions of averages. As an extension of equation (6.4) this section presents the decomposition of the change over time of differences and additions.

An area that has stimulated much interest among demographers has to do with gender differences. A few examples of these are the higher female survivorship in all populations, the higher male Mexican international migration, the higher female Mexican internal migration, the higher male mortality by violent causes of death, and the lower female wages in similar jobs as males.

The change over time of operations among demographic variables can be separated into direct effects and compositional effects. The chief interest is on the difference of demographic measures between the sexes, but other differences are possible. Let $\bar{v}_M(t)$ and $\bar{v}_F(t)$ denote two averages, for males and females respectively, and $\Delta\bar{v}(t)$ denote the difference between them, $\Delta\bar{v}(t) = \bar{v}_F(t) - \bar{v}_M(t)$. The change over time of the difference $\Delta\bar{v}(t)$ can be seen as

$$\dot{\Delta}\bar{v}(t) = \frac{\partial}{\partial t}\left[\bar{v}_F(t) - \bar{v}_M(t)\right] = \frac{\partial}{\partial t}\bar{v}_F(t) - \frac{\partial}{\partial t}\bar{v}_M(t) = \dot{\bar{v}}_F(t) - \dot{\bar{v}}_M(t). \tag{6.14}$$

The desired decomposition is achieved by applying equation (6.4) to both terms $\dot{\bar{v}}_F(t)$ and $\dot{\bar{v}}_M(t)$,

$$\begin{aligned}\dot{\Delta}\bar{v}(t) &= [\bar{\dot{v}}_F - \bar{\dot{v}}_M] + [C_F(v,\acute{w}) - C_M(v,\acute{w})] \\ &= \Delta\bar{\dot{v}} + \Delta C(v,\acute{w}). \end{aligned} \tag{6.15}$$

Table 6.3 presents the decomposition of the change over time of the difference of the Mexican male and female crude death rates. Using equation (6.15) the change in the difference of the crude death rates is explained by level-1 and level-2 effects of change. The periods studied are 1980-1985, 1985-1990 and 1990-1995. During the three studied periods the difference between male and female CDRs narrows. This can be seen in the negative values of the change in the difference, $\dot{\Delta}d(t)$. Between 1980 and 1985 the male-female mortality gap shows the greatest reduction. The level-1 effect is again the main contributor to this change. That is, the average reduction in male mortality is more effective than the female direct change. The two populations seem to have similar changes in age composition resulting in only a small difference due to the level-2 effect of change.

Table 6.3: Difference between the male and female crude death rates, $\Delta d(t)$, per thousand, and decomposition of the annual change over time between 1980-1985, 1985-1990 and 1990-1995 for Mexico.

t	1980		1985		1990		1995
$d_M(t)$	7.407		6.233		5.829		5.405
$d_F(t)$	5.517		4.755		4.394		4.112
$\Delta d(t)$	1.890		1.478		1.435		1.293
$\Delta \dot{d}(t)$		-0.082		-0.009		-0.028	
$\Delta\bar{\dot{\mu}}$		-0.088		-0.018		-0.037	
$\Delta C(\mu, r)$		0.005		0.009		0.009	
$\Delta \dot{d}(t) = \Delta\bar{\dot{\mu}} + \Delta C(\mu, r)$		-0.083		-0.009		-0.028	

Source: Author's calculations described in Chapter 9, based on the United Nations Data Base (2001). Note that $d_M(t)$ and $d_F(t)$ correspond to the male and female crude death rates.

The same formulations for the decomposition of a difference of averages could be developed for an addition of averages. As an example, let the reader be reminded of the expression for the crude birth rate in equation (4.15), in Chapter 4, as the addition of two CBRs,

$$CBR(t) = CBR_m(t) + CBR_u(t),$$

where $CBR_m(t)$ and $CBR_u(t)$ are the married and unmarried CBRs respectively. The change over time of (4.15) is decomposed in a similar way as in (6.15), i.e.,

$$C\dot{B}R(t) = C\dot{B}R_m(t) + C\dot{B}R_u(t). \tag{6.16}$$

In equations (4.13) and (4.16) it is shown that the crude birth rates of married and unmarried women is the product of three terms. For married women, for example, the three terms are the product of fertility of married women $b_{ma}(t)$, the proportion of married women $\pi_{ma}(t)$, and the proportion of women in the total population $\pi_{fa}(t)$. The change over time in this crude death rate is

$$C\dot{B}R_m(t) = \sum_0^{\omega} \dot{b}_{ma}(t)\pi_{ma}(t)\pi_{fa}(t)$$

$$\sum_0^{\omega} b_{ma}(t)\dot{\pi}_{ma}(t)\pi_{fa}(t) + \sum_0^{\omega} b_{ma}(t)\pi_{ma}(t)\dot{\pi}_{fa}(t). \tag{6.17}$$

Equation (6.17) shows the continuous expression for the difference in decomposition presented by Zeng et al. (1991) in (4.14). A similar decomposition is derived for the crude birth rate of unmarried women in (6.16). Here it should be noted that equation (6.17) has the advantage over (4.14) of not including residual terms and it is then an exact decomposition.

The change over time of other demographic functions could be studied. In the next two sections it is shown that the change over time of fractions and products of demographic averages can also be decomposed. The decomposition of these two types of operators is instead based on the relative change.

6.5 Relative Derivatives of Averages

Another formula presented by Vaupel (1992) is the decomposition of a relative change.

The relative change in an average, $\acute{\bar{v}}$, is

$$\acute{\bar{v}} = \frac{\dot{\bar{v}}(t)}{\bar{v}(t)} = \frac{1}{\bar{v}(t)} \frac{\partial}{\partial t} \frac{\int_0^\infty v(x,t)w(x,t)dx}{\int_0^\infty w(x,t)dx}. \tag{6.18}$$

The decomposition of the relative change over time of a demographic variable can be simply and memorably expressed as

$$\acute{\bar{v}} = \tilde{\acute{v}} + \left[\tilde{\acute{w}} - \bar{\acute{w}}\right], \tag{6.19}$$

where the tilde denotes an alternative averaging procedure where the weights are proportional to the product of $v(x,t)$ and $w(x,t)$,

$$\tilde{v}(t) = \frac{\int_0^\infty v(x,t)v(x,t)w(x,t)dx}{\int_0^\infty v(x,t)w(x,t)dx}. \tag{6.20}$$

The proof of this formula is directly established by dividing formula (6.4) by $\bar{v}$,

$$\acute{\bar{v}} = \frac{\dot{\bar{v}}}{\bar{v}} = \frac{\bar{\dot{v}}}{\bar{v}} + \frac{C(v, \acute{w})}{\bar{v}},$$

and from the definition of covariance in (2.13) we obtain

$$\acute{\bar{v}} = \tilde{\acute{v}} + \frac{\overline{v\acute{w}} - \bar{v}\bar{\acute{w}}}{\bar{v}} = \tilde{\acute{v}} + \left[\tilde{\acute{w}} - \bar{\acute{w}}\right]. \tag{6.21}$$

This formula separates the relative change into a level-1 and a level-2 change, as formula (6.4) did before.

Equation (6.19) could be applied in all the examples previously shown in this chapter, decomposing the relative change. As an example, we decompose the relative change of the crude death rate, $\bar{\mu}(t)$. This decomposition is obtained by applying equation (6.19) to the CDR,

$$\acute{\bar{\mu}} = \tilde{\acute{\mu}} + \left[r_D - r\right], \tag{6.22}$$

where $r(t)$ is the population growth rate and $r_D(t)$ represents the death-weighted population growth rate (i.e. the population growth rate when age categories are weighted by the number of deaths in that particular age category),

$$r_D(t) = \frac{\int_0^\infty r(a,t)\mu(a,t)N(a,t)da}{\int_0^\infty \mu(a,t)N(a,t)da} = \frac{\int_0^\infty r(a,t)D(a,t)da}{\int_0^\infty D(a,t)da}, \tag{6.23}$$

where $D(a,t)$ are the deaths occurring at age a and time t.

Equation (6.22) indicates that the relative change in the CDR is the result of two components. First, a level-1 change is seen in the average relative change in age-specific death rates.

Table 6.4: Crude death rate, $d(t)$, per thousand, and decomposition of the annual relative change over time in 1965-1975, 1975-1985 and 1985-1995 for Mexico.

t	1970	1980	1990
$\bar{\mu}(t)$	8.486	6.436	5.150
$\bar{\mu}(t-5)$	9.514	7.456	5.532
$\bar{\mu}(t+5)$	7.456	5.532	4.755
$\acute{\bar{\mu}}(t)$	-24.194	-29.774	-15.072
$\tilde{\acute{\mu}}$	-18.992	-24.583	-22.574
r_D	26.662	19.480	26.491
r	31.936	24.697	18.992
$r_D - r$	-5.275	-5.218	7.498
$\acute{\bar{\mu}}(t) = \tilde{\acute{\mu}} + [r_D - r]$	-24.267	-29.801	-15.076

Source: Author's calculations described in Chapter 9, based on the United Nations Data Base (2001).

Secondly, a compositional change is seen in the difference between the rise in deaths and the population growth.

Table 6.4 includes the Mexican crude death rate and the decomposition of its relative change during the periods 1965-1975, 1975-1985 and 1985-1995, denoted by their mid-years 1970, 1980 and 1990 respectively. Table 3.1, in Chapter 3, showed the Mexican crude death rate and the decomposition of the change over time. The results in the table for 1975-1985 contrast with those indicating relative change in Table 6.4. The relative change is more pronounced in the 1980s than in other periods.

In Table 6.4, the level-1 effect retains its predominance in the relative change. The level-2 effect is the result of the growth rate weighted by the deaths, $r_D(t)$, minus the growth rate of the population, $r(t)$. The growth rates that form the level-2 component are included in Table 6.4. The growth rate of the population decreases over time, falling from 3.1% to 1.8%. The growth rate weighted by the deaths decreases from 1970 to 1980 from 2.6% to 1.9% and later increases to 2.6% in 1990. As a result of the fluctuations in $r_D(t)$ there is a positive level-2 effect of change in 1990 that counters the decrease in the crude death rate. Increase in growth rates at ages with higher levels of mortality $\mu(a,t)$ produced this higher $r_D(t)$ with respect to $r(t)$ in 1990.

6.6 Relative Derivative of Products and Ratios

In Chapter 4 equation (4.25) showed the relative change of a product. If the function $v(t)$ is equal to the product of variables $v_i(t)$, $v(t) = v_1(t)v_2(t)...v_n(t)$, then the relative decomposition of $v(t)$ is the sum of the relative derivatives of $v_i(t)$,

$$\acute{v} = \acute{v}_1 + \acute{v}_2 + ... + \acute{v}_n.$$

This formula is a particular case of the general formula of an average. Let the demographic variable $v(a,t)$ be a product of variables $v(a,t) = v_1(a,t)v_2(a,t)...v_n(a,t)$. Let the demographic function $w(a,t)$ be a product of variables $w(a,t) = w_1(a,t)w_2(a,t)...w_m(a,t)$. The average $\bar{v}(t)$ is now expressed as follows,

$$\begin{aligned}\bar{v}(t) &= \frac{\int_0^\omega v\,(a,t)\,w\,(a,t)\,da}{\int_0^\omega w\,(a,t)\,da} \\ &= \frac{\int_0^\omega v_1(a,t)v_2(a,t)...v_n(a,t)w_1(a,t)w_2(a,t)...w_m(a,t)da}{\int_0^\omega w_1(a,t)w_2(a,t)...w_m(a,t)da}. \end{aligned} \tag{6.24}$$

From equation (6.19), for the decomposition of the relative change, and from equation (4.25), the relative change of a product, we obtain the extension of the decomposition

$$\begin{aligned}\acute{\bar{v}} &= \tilde{\bar{v}}_1 + \tilde{\bar{v}}_2 + ... + \tilde{\bar{v}}_n \\ &\quad + \left[\tilde{\bar{w}}_1 - \bar{\bar{w}}_1\right] + \left[\tilde{\bar{w}}_2 - \bar{\bar{w}}_2\right] ... + \left[\tilde{\bar{w}}_m - \bar{\bar{w}}_m\right]. \end{aligned} \tag{6.25}$$

The first n components are the level-1 effects of the n variables $v_1(a,t)v_2(a,t)...v_n(a,t)$. The second group of components are the level-2 effects of the m variables $w_1(a,t)w_2(a,t)...w_m(a,t)$.

Take, for example, the crude birth rate presented in equation (2.8). As mentioned in Chapter 2, the crude birth rate can be seen as an average by using the female ratio, $c_f(a,t) = \frac{N_f(a,t)}{N(a,t)}$,

$$CBR(t) = \frac{\int_0^\omega b\,(a,t)\,c_f\,(a,t)\,N\,(a,t)\,da}{\int_0^\omega N\,(a,t)\,da}.$$

Applying equation (6.25) to the CBR it is possible to decompose the relative change of the crude birth rate as

$$C\acute{B}R = \tilde{\bar{b}} + \tilde{\bar{c}}_f + [r_B - r], \tag{6.26}$$

where $r(t)$ and $r_B(t)$ are the population growth rates and the births of the weighted population growth rate (i.e. the population growth rate when age categories are weighted by the number of births to women in that particular age category). Hence (6.26) provides a breakdown of the growth rate of the crude birth rate into components related to the growth rate of the age-specific fertility rates, the growth rate of the female ratio and the degree of disproportionate population growth at fertile ages.

Tables 4.3 and 4.4 included the crude birth rate and the decomposition of the change over time for Denmark, the Netherlands and Sweden. Table 6.5 presents the decomposition of the relative change in the CBR seen in (6.26) for these same countries from 1992-1997, with 1995 the mid-year.

Complementary results obtained by looking at Tables 4.3 and 4.4 are further established by observing Table 6.5. The average relative change in birth rates, $\tilde{\bar{b}}$, is negative for all countries. In other words, on average there is a decrease in the age-specific birth rates. This level-1 effect of change shows an enormous reduction and is the main contributor to the relative change in Sweden.

The level-2 change, a result of the difference in growth rates, is the main contributor to the relative change in the CBR for Denmark and the Netherlands. In the latter country, the

Table 6.5: Crude birth rate, $CBR(t)$, in percentage, and decomposition of the annual relative change over time, in percentage, from 1992 to 1997, for Denmark, the Netherlands and Sweden.

	Denmark	Netherlands	Sweden
$CBR(1995)$	1.287	1.258	1.199
$CBR(1992)$	1.310	1.295	1.417
$CBR(1997)$	1.280	1.232	1.023
$C\acute{B}R(1995)$	-0.464	-0.996	-6.548
$\tilde{\acute{b}}$	-0.121	-0.285	-6.238
$\tilde{\acute{c}}_f$	0.087	0.060	0.073
r_B	0.006	-0.212	0.055
r	0.434	0.554	0.408
$r_B - r$	-0.428	-0.766	-0.353
$C\acute{B}R = \tilde{\acute{b}} + \tilde{\acute{c}}_f + [r_B - r]$	-0.462	-0.991	-6.518

Source: Author's calculations described in Chapter 9, based on Eurostat (2000).

growth rate weighted by babies is negative as a result of negative growth rates in the ages where many children were born.

Equation (6.25) can also be extended to include products and ratios of averages. If the average $\bar{v}(t)$ is the product of averages, $\bar{v}(t) = \bar{v}_1(t)\bar{v}_2(t)...\bar{v}_n(t)$, then the decomposition of the relative change is similar to (6.25). The ratios have to be expressed in another way.

Let $\bar{v}_M(t)$ and $\bar{v}_F(t)$ be the averages for males and females, and the ratio of males over females be $\bar{v}(t) = \frac{\bar{v}_M(t)}{\bar{v}_F(t)}$. This measure is decomposed as follows

$$\acute{\bar{v}} = \tilde{\acute{v}}_M - \tilde{\acute{v}}_F(t) + \left[\tilde{\acute{w}}_M - \bar{\acute{w}}_M\right] - \left[\tilde{\acute{w}}_F - \bar{\acute{w}}_F\right]. \tag{6.27}$$

The proof follows from equation (6.25) by changing the ratio for a product $\bar{v}(t) = \bar{v}_M(t)\left[\bar{v}_F(t)\right]^{-1}$ derived with respect to t.

The ratio of the total fertility rate (TFR) of married over unmarried can be calculated as

$$R_{TFR}(t) = \frac{TFR_m(t)}{TFR_u(t)} = \frac{\int_\alpha^\beta b_m(a,t)da}{\int_\alpha^\beta b_u(a,t)da}, \tag{6.28}$$

where $b_m(a,t)$ and $b_u(a,t)$ are, as before, the age-specific fertility rates of married and unmarried women. The decomposition of the relative change can be calculated following (6.27) which yields

$$\acute{R}_{TFR} = T\acute{F}R_m(t) - T\acute{F}R_u(t), \tag{6.29}$$

where the relative change in $T\acute{F}R$ is equal to the average relative change in fertility rates weighted by the fertility rates, for example for the married population we have

$$\acute{TFR}_m(t) = \frac{\int_\alpha^\beta \acute{b}_m(a,t) b_m(a,t) da}{\int_\alpha^\beta b_m(a,t) da}. \tag{6.30}$$

A similar ratio is used by Coale (mentioned by Preston et al. (2001)) to study the historical fertility levels in European populations. Table 6.6 presents the ratio of TFR for those married over those unmarried and the decomposition shown in equation (6.29) for the relative change from 1992 to 1997 applied to data from Denmark, the Netherlands and Sweden. The gap

Table 6.6: Ratio of total fertility rates for married over unmarried women and the decomposition of the annual relative change over time, in percentage, for Denmark, the Netherlands and Sweden from 1992 to 1997.

	Denmark	Netherlands	Sweden
$R_{TFR}(1995)$	2.860	6.938	3.269
$R_{TFR}(1992)$	2.796	7.712	3.237
$R_{TFR}(1997)$	2.925	6.242	3.301
$\acute{R}_{TFR}(1995)$	0.009	-0.042	0.004
$\acute{TFR}_m(t)$	-0.240	-0.558	-0.287
$\acute{TFR}_u(t)$	-0.251	-0.488	-0.293
$\acute{R}_{TFR} = \acute{TFR}_m(t) - \acute{TFR}_u(t)$	0.011	-0.070	0.006

Source: Author's calculations from formula (6.29), based on Eurostat (2000).

between married and unmarried TFR is smaller in the Scandinavian countries than in the Netherlands. All three countries experienced a decline in the average relative change in age-specific fertility rates. For both Denmark and Sweden, the fertility of the unmarried exhibits a more pronounced decrease compared to the married. As a result both have positive relative changes in the ratio while there is a negative change for the Netherlands.

6.7 Decomposition of the Change in Life Expectancy

6.7.1 The Derivative of Life Expectancy

Vaupel and Canudas Romo (2003) have shown that life expectancy can also be decomposed into two components analogous to those in equation (6.4). These terms account for the mortality improvements and the heterogeneity of such improvements at different ages.

The notation of life expectancy introduced in Chapter 4 is used here, together with some new notations. Let $f(a,t)$ denote the probability density function describing the distribution of deaths (i.e., lifespan) in the lifetable population at age a and at time t. This function is equal to the product of force of mortality and the survival function, $f(a,t) = \mu(a,t)\ell(a,t)$.

As already defined, the rate of progress in reducing mortality rates is $\rho(a,t) = -\acute{\mu}(a,t)$. The average improvement in mortality is calculated as

$$\bar{\rho}(t) = \int_0^\omega \rho(a,t)f(a,t)da. \tag{6.31}$$

Let $e^o(a,t)$ denote remaining life expectancy at age a and time t:

$$e^o(a,t) = \frac{\int_a^\omega \ell(x,t)dx}{\ell(a,t)}, \tag{6.32}$$

the average number of life-years lost as a result of death is

$$e^\dagger(t) = \int_0^\omega e^o(a,t)f(a,t)da. \tag{6.33}$$

Recalling the definition of covariance in (2.12), the covariance between the rate of improvement in mortality and remaining life expectancy at various ages is

$$C_f(\rho, e^o) = \int_0^\omega \left[\rho(a,t) - \bar{\rho}(t)\right]\left[e^o(a,t) - e^\dagger(t)\right] f(a,t)da. \tag{6.34}$$

Note that the averages and the covariance just introduced have denominators of one because the probability density function describing the distribution of deaths over all ages is equal to one, $\int_0^\omega f(a,t)da = 1$.

Vaupel and Canudas Romo's main formula for decomposing the change in life expectancy is

$$\dot{e}^o(0,t) = \bar{\rho}e^\dagger + C_f(\rho, e^o). \tag{6.35}$$

The proof follows from substituting equations (6.31), (6.33) and (6.34) into (6.35). Another demonstration is found in Vaupel and Canudas Romo (2003).

The terms in equation (6.35) are on one side of the equation the derivative of life expectancy at birth, $\dot{e}^o(0,t)$, that is the change in life expectancy over time. The right hand side of the formula contains two terms. The first term is the product of the average rate of mortality improvement and the average number of life-years lost. The average rate of mortality improvement $\bar{\rho}(t)$ can be interpreted as the proportion of deaths averted (or lives saved), while $e^\dagger(t)$ can be interpreted as the average number of life-years gained per life saved.

The second term, the covariance between rates of mortality improvement and remaining life expectancies, increases or decreases the general effect, depending on whether the covariance is positive or negative. If $\rho(a,t)$ is constant at all ages, then the covariance is zero. Hence, the covariance captures the effect of heterogeneity in $\rho(a,t)$ at different ages. Remaining life expectancy generally declines with age, as shown for three countries in Figure 6.1. At a certain age a^*, the remaining life expectancy is equal to the average number of life-years gained per life saved, that is $e^o(a^*,t) = e^\dagger(t)$. The covariance will be positive if before age a^* the age-specific pace of mortality improvement tends to be higher than average and if after age a^* the age-specific pace of mortality improvement tends to be lower than average. Note that ages are weighted according to the distribution of deaths.

Table 6.7: Life expectancy at birth, $e^o(0,t)$, and the decomposition of the annual change from 1990 to 1999 for Japan, Sweden and the United States.

	Japan	*Sweden*	*United States*
$e^o(0,1995)$	79.821	78.551	76.094
$e^o(0,1990)$	78.983	77.591	75.441
$e^o(0,1999)$	80.658	79.512	76.746
$\dot{e}^o(0,1995)$	0.186	0.213	0.145
$\bar{\rho}$ (%)	2.202	1.426	0.537
$e^\dagger$	10.418	10.253	12.399
$\bar{\rho}e^\dagger$	0.229	0.146	0.067
$C_f(\rho, e^o)$	-0.044	0.066	0.078
$\dot{e}^o(0,1995) = \bar{\rho}e^\dagger + C_f(\rho, e^o)$	0.186	0.213	0.145

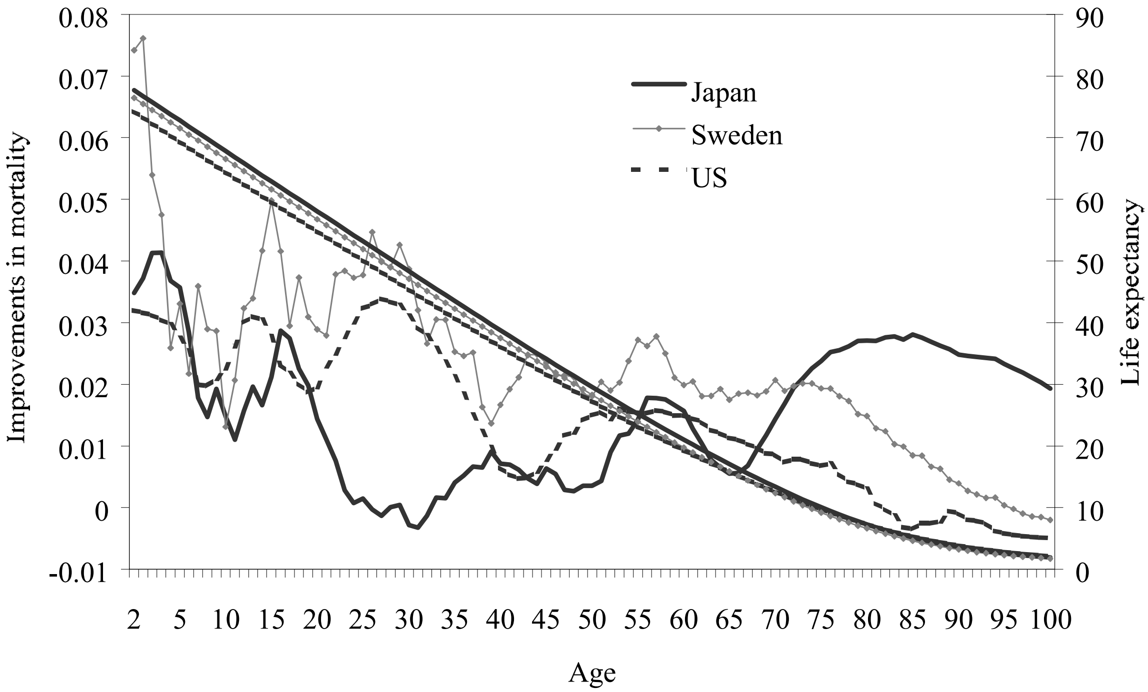
Source: Author's calculations are described in Chapter 9. Lifetable data is derived from the Human Mortality Database (2002). Lifetable values from the years 1990 and 1999 were used to obtain results for the mid-point around January 1995.

Table 6.7 shows the application of equation (6.35) to the annual change in life expectancy at birth for Japan, Sweden and the United States from 1990 to 1999, on January 1, 1995. In the last decade of twentieth century, Japan, Sweden and the United States all experienced increasing levels of life expectancy. Particularly interesting are the female Japanese levels, which are world records. Oeppen and Vaupel (2002) presented a complete list of these world records in the last century and an explanation of how the forecasted limits of life expectancy have been broken. The number of life-years lost as a result of death, $e^\dagger(t)$, varied for each country, with Japan registering 10.41 years, Sweden 10.25 years and the US 12.40 years. These numbers are multiplied by the improvements in mortality, $\bar{\rho}(t)$. Japan exhibits the greatest proportion of prolonged lives saved whereas the US has the lowest proportion. As a consequence, the product $\bar{\rho}(t)e^\dagger(t)$ gives the term of the contribution in life expectancy due to the advance in survivorship. For Japan and the US, this term—which is the main contributor to the increase in life expectancy—is positive.

The second term is the covariance between the improvements in mortality and the remaining life expectancy. Sweden and the United States share similar levels of this component. Nevertheless, for the US this is the most important factor. Figure 6.1 shows the five-year moving average of the improvement in mortality and the remaining life expectancy for ages 2 to 100 for the three countries. In this figure, it is possible to see that the remaining life expectancy generally declines with age. The improvement in mortality underwent numerous fluctuations. The five-year moving average is shown here. The positive covariance between these two measures results from major improvements in mortality in the younger age groups where life expectancy is the highest. Furthermore, in Sweden and the US the curves of improvement in both mortality and remaining life expectancy decline with age.

In Japan the covariance is in opposition to the increase in life expectancy. This is the result of a change in the curve of improvement in mortality around 60 shifting up in the oldest age groups. This covariance term could be divided into positive values before reaching the sixties

Figure 6.1: Five-year moving average of the improvement in mortality and the remaining life expectancy at ages 2 to 100 for Japan, Sweden and the United States in 1995.



and negative values there after. How this age decomposition can be achieved, as well as, a cause of death decomposition is shown in Chapter 7.

6.7.2 The Difference in Male and Female Life Expectancy

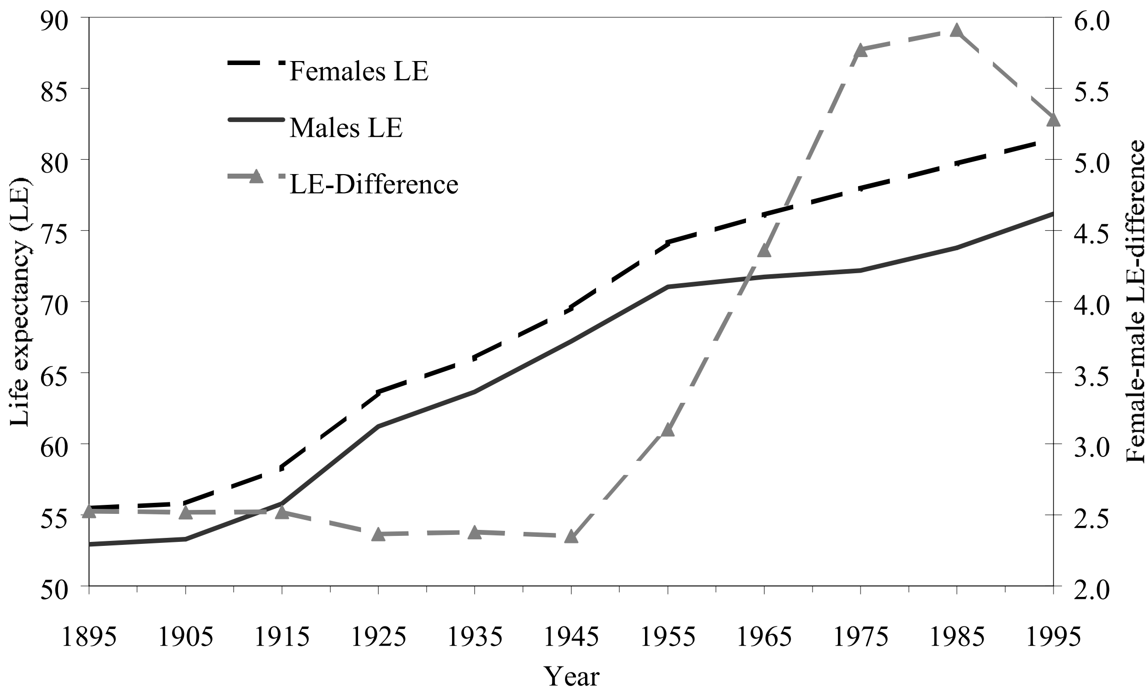
It is well known that a substantial gap exists between the life expectancy of males and females. Retherford (1972), Pollard (1982 and 1988), Van Poppel, Tabeau and Willekens (1996), and Valkovics (in Wunsch (2002)) are some of the studies that look at the mortality differential between the sexes. An interesting question, therefore, concerns how this difference has changed over time.

Let $e^o_F(0,t)$ and $e^o_M(0,t)$ be the life expectancy at birth for females and males respectively. The change over time in the difference between the female-male life expectancies is

$$\frac{\partial}{\partial t}\left[e^o_F(0,t) - e^o_M(0,t)\right] = \dot{e}^o_F - \dot{e}^o_M. \tag{6.36}$$

In equation (6.36) any of the decompositions of life expectancy mentioned in Chapter 4 could be substituted. For an example we look at the decomposition of the change in the difference of the female-male life expectancy in Sweden for every decade between 1895 and 1995. Figure 6.2 has two vertical axes, the left corresponding to life expectancies and the right to the differences, both in years. Until 1945 the difference was around 2.5 years, with an increase after this year until the peak in 1985, followed by a reduction in the gap in the latest years. The decomposition of equation (6.35) is applied to study the difference of the female-male life

Figure 6.2: Life expectancy for Swedish males and females from 1895 to 1995, and the difference in life expectancies.



expectancy in Sweden. For every year an average change and a covariance term is obtained

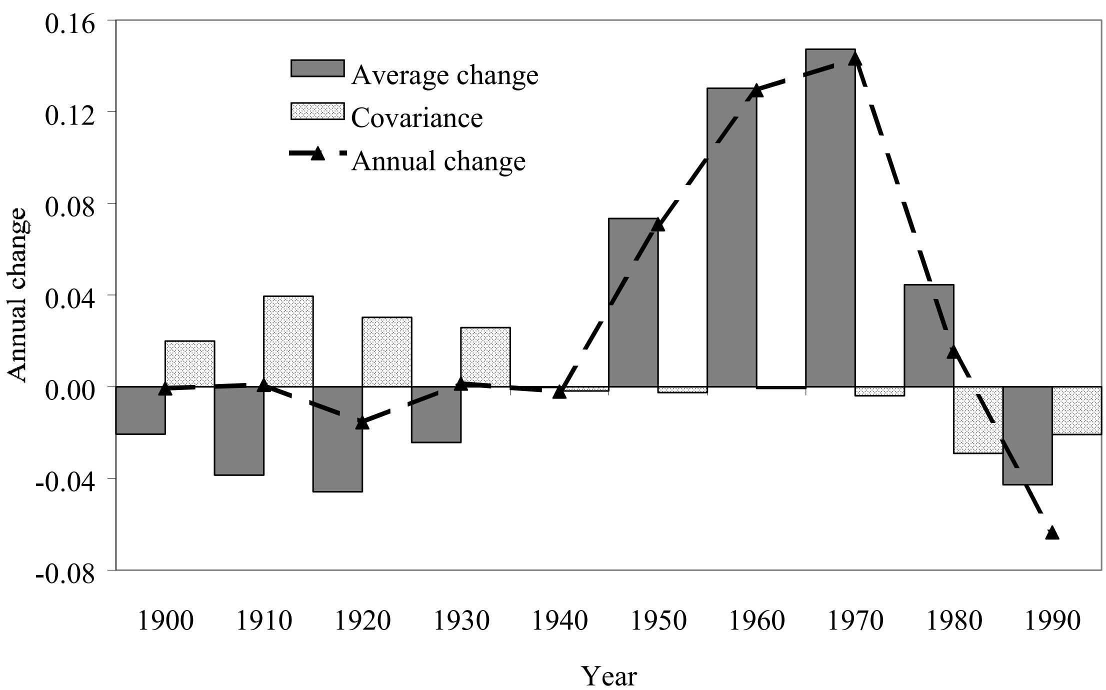
$$\begin{aligned}\frac{\partial}{\partial t}\left[e_F^o(0,t) - e_M^o(0,t)\right] &= \left[\bar{\rho}e^\dagger\right]_F - \left[\bar{\rho}e^\dagger\right]_M + \left[C_f(\rho, e^o)\right]_F - \left[C_f(\rho, e^o)\right]_M \\ &= \left[\bar{\rho}e^\dagger\right]_{F-M} + \left[C_f(\rho, e^o)\right]_{F-M}. \end{aligned} \tag{6.37}$$

Figure 6.3 shows the result of applying equation (6.37) to obtain the two components of change over time. Three periods can be noted here: before 1940, from 1940 to 1970, and after 1970. In the period before 1940 the advance in survivorship, $\bar{\rho}e^\dagger$, was better for males than females and the difference was negative. This tendency reverts between 1940-1970 when the females took the lead on advancements in survivorship. During the last period the rate of male survivorship outpaced female survivorship. Similar explanations can be reached for the covariance component.

6.8 Conclusion

In this chapter we present and prove various formulas for decomposing change in a population average into components.

Figure 6.3: Decomposition of the annual change in the Swedish male and female life expectancy difference from 1895 to 1995.



The direct vs. compositional decomposition requires notation that has not become completely standard in demography for three terms, namely the average, derivative and relative derivative, introduced in Chapter 2. Once these notations are adopted we obtain a simple and elegant equation (6.4).

Equation (6.4) provides a straightforward but powerful approach to decomposing direct versus compositional changes for many applications. One component captures the effect of direct change in the characteristic of interest, and the other captures the change that is attributable to a change in the structure or composition of the population.

The decomposition is applied to time derivatives of averages over age and over subpopulations. Several illustrative examples are provided in this chapter. In the examples shown here, two kinds of compositional changes are studied: changes in the age structure of the population and in the size of subpopulations. Many other applications are possible. Such applications will help demographers understand the dynamics of population change.

Equation (6.19) provides a decomposition of the intensity of averages. This alternative way of studying the dynamic of demographic averages is also separated into two components. The first is the average intensity of change of the variable of interest. The second is a difference of averages of the relative change in weights. This second component is analogous to the covariance in equation (6.4) accounting for the change attributed to the change in the structure of the population.

Equation (6.35) provides a new decomposition of the change in life expectancy. The method permits decomposition of the change in life expectancy into the general impact of mortality improvement at all ages and the additional effect of heterogeneity on the age-specific rates of improvement. Vaupel and Canudas Romo (2003) also show that this method permits further decomposition of age-specific and cause-specific effects —for each age category or for each cause of death—of the effects of the pace of mortality improvement, remaining life expectancy, and the frequency of deaths.

The decompositions introduced here, (6.4), (6.19) and (6.35), will lead to interesting demographic insights on population dynamics. The streamlined formulas permit deeper comprehension of the demographic factors that drive changes in demographic variables. Although it is a minor drawback, it is necessary to use approximations when applying them to data, as explained in Chapter 9.